DOCUMENT RESUME

ED 092 587 TM 003 698

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TITLE Multivariate Profile Analysis of Split-Split Plot Designs and Growth Curve Analysis of Multivariate Repeated Measures Designs.
REPORT NO WP-29
PUB DATE 74
NOTE 72p.; Paper presented at the Annual Meeting of the American Educational Research Association (Chicago, Illinois, April, 1974)

EDRS PRICE MF-$0.75 HC-$3.15 PLUS POSTAGE
DESCRIPTORS *Hypothesis Testing; Matrices; *Models; *Nonparametric Statistics; Profile Evaluation; *Statistical Analysis; *Tests of Significance
IDENTIFIERS Repeated Measures Design

ABSTRACT

Multivariate models are demonstrated to analyze repeated measures profile and growth curve data when univariate or multivariate mixed model assumptions are not tenable. Standard mixed model tests are recovered from certain multivariate hypotheses. The procedures are illustrated using numerical examples. (Author/RC)

Working Paper No. 29

of the

Department of Educational Research

Multivariate Profile Analysis of Split-Split Plot Designs and Growth Curve Analysis of Multivariate Repeated Measures Designs*

Neil H. Timm

*Paper presented at the annual meeting of the American Educational Research Association, Chicago, Illinois, April 1974.

Multivariate Profile Analysis of Split-Split Plot Designs and Growth Curve Analysis of Multivariate Repeated Measures Designs[1]

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I. Introduction

A classical univariate mixed model analysis of a split-plot design as discussed for example by Geisser and Greenhouse (1958), Greenhouse and Geisser (1959), Kirk (1968, p. 250), Myers (1966, p. 223) and Winer (1971, p. 250), among others, is familiar to most experimenters. The classical mixed linear model for the design is

$$y_{ijk} = \mu+\alpha_i+\beta_k+(\alpha\beta)_{ik}+s_{(i)j}+\varepsilon_{(i)jk} \tag{1}$$

where,

- μ is an arbitrary constant,
- α_i is the effect of the i^{th} treatment group which is constant for all subjects within treatment group i,
- β_k is the effect of the k^{th} profile condition for all subjects,
- $(\alpha\beta)_{ik}$ is the interaction effect of α_i and β_k,
- $s_{(i)j}$ is the component associated with subject j, nested within α_i, and
- $\varepsilon_{(i)jk}$ is the error component for subject j within treatment group i for the k^{th} profile condition.

In addition, the random components $s_{(i)j}$ and $\varepsilon_{(i)jk}$ are assumed to be jointly independent and normally distributed:

$$s_{(i)j} \sim IN(0, \rho\sigma^2)$$

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$$\epsilon_{(i)jk} \sim IN(0, (1-\rho)\sigma^2)$$

Thus, the variance-covariance matrix Σ has the form

$$\Sigma = \rho\sigma^2 J + (1-\rho)\sigma^2 I = \sigma_s^2 J + \sigma_e^2 I$$

where J is a matrix of unities and I is an identity matrix so that Σ satisfies the compound symmetry assumption. Although the compound symmetry assumption is only a sufficient condition for exact univariate F-tests and not a necessary condition, as shown by Huynh and Feldt (1970), to analyze split-plot designs, it is the one most often assumed by researchers. When Σ is not of the appropropriate form for a univariate mixed model analysis, a multivariate analysis of the data is most appropriate. Koch (1969) reviewed the parametric and nonparametric multivariate analysis procedures one should use to analyze the split-plot design given in (1) using the classical multivariate linear model for arbitrary Σ under normality and nonnormality. More recently, Timm and Carlson (1973) demonstrated, following Bock (1963a, 1963b), the correspondence between univariate and numerous multivariate tests employing a multivariate full rank linear model. In this paper we extend the work of Timm and Carlson (1973) to split-split plot designs and also show how the growth curve model introduced by Potthoff and Roy (1964) and studied by Khatri (1966), Rao (1965, 1966, 1967), Grizzle and Allen (1969) and Kleinbaum (1973a), among others, may be used to analyze split-plot and split-split plot designs with multivariate repeated measures.

2. Analyzing Split-Plot Designs

Before discussing split-split plot designs, it is convenient to introduce some notation involving the analysis of split-plot designs and to review the correspondence between a univariate and multivariate analysis of these designs. The adaptation of the agricultural split-plot design to the behavioral sciences may be separated into two categories; split-plot profile analysis repeated measures designs and split-plot trend analysis repeated measures designs. Using Figure 1 to make the distinction between the two types of split-plot repeated measures designs, we say that we have a profile analysis if the levels of B are not ordered and a trend analysis if the levels of B are ordered. The s_i in Figure 1 indicate that subjects are measured repeatedly over all levels of factor B, but that the subjects within each level of A are different.

B (Conditions)

		B_1	B_2	B_3	B_4
A (groups)	A_1	s_1	s_1	s_1	s_1
	A_2	s_2	s_2	s_2	s_2

Figure 1. Split-Plot Repeated Measures Design

Since Timm and Carlson (1973) only considered the correspondence between univariate and multivariate analysis of repeated measures profile data, we shall briefly review the correspondence between univariate and multivariate analysis of both types of data. For this purpose, the data given in Table 1, from Kirk (1968, p. 274), are reanalyzed. The univariate profile and trend analysis of the data are displayed in Table 2 and Table 3, respectively. For either analysis the scores associated with factor B are assumed to be commensurable, expressed in the same units.

Table 1. Kirk's Data

		B_1	B_2	B_3	B_4
A_1	s_1	3	4	7	7
	s_2	6	5	8	8
	s_3	3	4	7	9
	s_4	3	3	6	8
A_2	s_1'	1	2	5	10
	s_2'	2	3	6	10
	s_3'	2	4	5	9
	s_4'	2	3	6	11

Table 2. Univariate Profile Analysis

Hypotheses	SS	DF	MS	F	p-value
1. Constant	924.50	1	924.50	$(\frac{1}{3}) = 592.63$	< .0001
2. A	3.13	1	3.13	$(\frac{2}{3}) = 2.00$	0.2070
3. S(A)	9.38	6	1.56		
4. B	194.50	3	64.83	$(\frac{4}{6}) = 127.88$	< .0001
5. AB	19.38	3	6.46	$(\frac{5}{6}) = 12.74$	0.0001
6. Error	9.13	18	0.51		
7. Total	1160.00	32			

Table 3. Univariate Trend Analysis

Hypotheses	SS	DF	MS	F	p-value
1. Constant	924.50	1	924.50	$(\frac{1}{3}) = 592.63$	<.0001
2. A	3.13	1		$(\frac{2}{3}) = 2.00$	0.2070
3. S(A)	9.38	6			
4. B	194.50	3	64.83		
5. Linear trend	184.90	1	184.90	$(\frac{5}{13}) = 182.71$	<.0001
6. Quadratic trend	8.00	1	8.00	$(\frac{6}{14}) = 25.64$	0.00[illegible]
7. Cubic trend	1.60	1	1.60	$(\frac{7}{15}) = 8.16$	0.0289
8. AB	19.38	3	6.46		
9. Linear trend	13.23	1	13.23	$(\frac{9}{13}) = 13.07$	0.0112
10. Quadratic trend	3.13	1	3.13	$(\frac{10}{14}) = 10.02$	0.0195
11. Cubic trend	3.03	1	3.03	$(\frac{11}{15}) = 15.43$	0.0077
12. Error	9.13	18	0.51		
13. Linear trend	6.08	6	1.01		
14. Quadratic trend	1.88	6	0.31		
15. Cubic trend	1.18	6	.20		
16. Total	1160.00	32			

Using the restricted full rank linear model, as developed by Timm and Carlson (1973, 1974a, 1974b) to represent the classical split-plot repeated measures design, (1), the restricted full rank linear model for the design is

(2) $$y_{ijk} = \mu_{ijk} + \epsilon_{(i)jk}$$

$i=1, \ldots, I;\ j=1, \ldots, N_i;\ k=1, \ldots, p$

subject to $\sum_i (N_i-1)(p-1) = (N-I)(p-1)$ linearly independent

restrictions

$$\mu_{ijk}-\mu_{ij'k}-\mu_{ijk'}+\mu_{ij'k'} = 0$$

where i is the treatment group index, j is the subject index, with subjects nested within groups, and k is the repeated measures index over p profile conditions. For a trend analysis the restrictions in (2) are modified, Table 6. The hypotheses being tested in ANOVA Table 2 are shown in Table 4. The population means associated with Kirk's data are given in Table 5 where the familiar dot notation is used to represent averages.

Table 4. Univariate Profile Analysis Hypotheses

Source	Hypotheses	DF
Constant	$\mu_{...} = 0$	1
A	all $\mu_{i..}$'s are equal	I-1
S(A)	$\mu_{11.} = \mu_{12.} = \cdots = \mu_{1N_1.}$ $\vdots$ $\mu_{I1.} = \mu_{I2.} = \cdots = \mu_{IN_I.}$	N-I
B	all $\mu_{..k}$'s are equal	p-1
AB	$\mu_{i.k}-\mu_{i'.k}-\mu_{i.k'}+\mu_{i'.k'} = 0$	(I-1)(p-1)
Error	$\mu_{ijk}-\mu_{ij'k}-\mu_{ijk'}+\mu_{ij'k'} = 0$	(N-I)(p-1)
Total		Np

Table 5. Population Parameters for Kirk's Data

		B_1	B_2	B_3	B_4	(Means)
A_1	s_1	μ_{111}	μ_{112}	μ_{113}	μ_{114}	$\mu_{11.}$
	s_2	μ_{121}	μ_{122}	μ_{123}	μ_{124}	$\mu_{12.}$
	s_3	μ_{131}	μ_{132}	μ_{133}	μ_{134}	$\mu_{13.}$
	s_4	μ_{141}	μ_{142}	μ_{143}	μ_{144}	$\mu_{14.}$
	(Means)	$\mu_{1.1}$	$\mu_{1.2}$	$\mu_{1.3}$	$\mu_{1.4}$	$\mu_{1..}$
A_2	s'_1	μ_{211}	μ_{212}	μ_{213}	μ_{214}	$\mu_{21.}$
	s'_2	μ_{221}	μ_{222}	μ_{223}	μ_{224}	$\mu_{22.}$
	s'_3	μ_{231}	μ_{232}	μ_{233}	μ_{234}	$\mu_{23.}$
	s'_4	μ_{241}	μ_{242}	μ_{243}	μ_{244}	$\mu_{24.}$
	(Means)	$\mu_{2.1}$	$\mu_{2.2}$	$\mu_{2.3}$	$\mu_{2.4}$	$\mu_{2..}$
		$\mu_{..1}$	$\mu_{..2}$	$\mu_{..3}$	$\mu_{..4}$	$\mu_{...}$

Using the means in Table 5, the hypotheses tested in Table 3 are summarized in Table 6; those sources not shown in Table 6 are identical to the expressions given in Table 4.

Table 6. Univariate Trend Analysis Hypotheses

Source	Hypotheses	DF
B		
Linear trend	$-3\mu_{..1}-\mu_{..2}+\mu_{..3}+3\mu_{..4}=0$	1
Quadratic trend	$\mu_{..1}-\mu_{..2}-\mu_{..3}+\mu_{..4}=0$	1
Cubic trend	$-\mu_{..1}+3\mu_{..2}-3\mu_{..3}+\mu_{..4}=0$	1
AB		
Linear trend	$-3\mu_{1.1}-\mu_{1.2}+\mu_{1.3}+3\mu_{1.4}+3\mu_{2.1}+\mu_{2.2}-\mu_{2.3}-3\mu_{2.4}=0$	1
Quadratic trend	$\mu_{1.1}-\mu_{1.2}-\mu_{1.3}+\mu_{1.4}-\mu_{2.1}+\mu_{2.2}+\mu_{2.3}-\mu_{2.4}=0$	1
Cubic trend	$-\mu_{1.1}+3\mu_{1.2}-3\mu_{1.3}+\mu_{1.4}+\mu_{2.1}-3\mu_{2.2}+3\mu_{2.3}-\mu_{2.4}=0$	1
Error		
Linear (Restrictions)	$-3\mu_{111}-\mu_{112}+\mu_{113}+3\mu_{114}+3\mu_{121}+\mu_{122}-\mu_{123}-3\mu_{124}=0$	
	$-3\mu_{121}-\mu_{122}+\mu_{123}+3\mu_{124}+3\mu_{131}+\mu_{132}-\mu_{133}-3\mu_{134}=0$	
	$-3\mu_{131}-\mu_{132}+\mu_{133}+3\mu_{134}+3\mu_{141}+\mu_{142}-\mu_{143}-3\mu_{144}=0$	6
	$-3\mu_{211}-\mu_{212}+\mu_{213}+3\mu_{214}+3\mu_{221}+\mu_{222}-\mu_{223}-3\mu_{224}=0$	
	$-3\mu_{221}-\mu_{222}+\mu_{223}+3\mu_{224}+3\mu_{231}+\mu_{232}-\mu_{233}-3\mu_{234}=0$	
	$-3\mu_{231}-\mu_{232}+\mu_{233}+3\mu_{234}+3\mu_{241}+\mu_{242}-\mu_{243}-3\mu_{244}=0$	
Quadratic (Restrictions)	$\mu_{111}-\mu_{112}-\mu_{113}+\mu_{114}-\mu_{121}+\mu_{122}+\mu_{123}-\mu_{124}=0$	
	$\mu_{121}-\mu_{122}-\mu_{123}+\mu_{124}-\mu_{131}+\mu_{132}+\mu_{133}-\mu_{134}=0$	
	$\mu_{131}-\mu_{132}-\mu_{133}+\mu_{134}-\mu_{141}+\mu_{142}+\mu_{143}-\mu_{144}=0$	6
	$\mu_{211}-\mu_{212}-\mu_{213}+\mu_{214}-\mu_{221}+\mu_{222}+\mu_{223}-\mu_{224}=0$	
	$\mu_{221}-\mu_{222}-\mu_{223}+\mu_{224}-\mu_{231}+\mu_{232}+\mu_{233}-\mu_{234}=0$	
	$\mu_{231}-\mu_{232}-\mu_{233}+\mu_{234}-\mu_{241}+\mu_{242}+\mu_{243}-\mu_{244}=0$	
Cubic (Restrictions)	$-\mu_{111}+3\mu_{112}-3\mu_{113}+\mu_{114}+\mu_{121}-3\mu_{122}+3\mu_{123}-\mu_{124}=0$	
	$-\mu_{121}+3\mu_{122}-3\mu_{123}+\mu_{124}+\mu_{131}-3\mu_{132}+3\mu_{133}-\mu_{134}=0$	
	$-\mu_{131}+3\mu_{132}-3\mu_{133}+\mu_{134}+\mu_{141}-3\mu_{142}+3\mu_{143}-\mu_{144}=0$	6
	$-\mu_{211}+3\mu_{212}-3\mu_{213}+\mu_{214}+\mu_{221}-3\mu_{222}+3\mu_{223}-\mu_{224}=0$	
	$-\mu_{221}+3\mu_{222}-3\mu_{223}+\mu_{224}+\mu_{231}-3\mu_{232}+3\mu_{233}-\mu_{234}=0$	
	$-\mu_{231}+3\mu_{232}-3\mu_{233}+\mu_{234}+\mu_{241}-3\mu_{242}+3\mu_{243}-\mu_{244}=0$	

Following Timm and Carlson (1974a, 1974b), (2) is written as follows:

(3)
$$\Omega: \underset{N \times 1}{\underset{\sim}{y}} = \underset{N \times p}{W} \underset{p \times 1}{\underset{\sim}{\mu}} + \underset{N \times 1}{\underset{\sim}{\varepsilon}}$$

subject to a set of restrictions

$$\underset{s \times p}{R'} \underset{p \times 1}{\underset{\sim}{\mu}} = \underset{s \times 1}{\underset{\sim}{0}}$$

The form of the hypotheses in Tables 4 and 6 is

(4)
$$H: \underset{\nu_h \times p}{C'} \underset{p \times 1}{\underset{\sim}{\mu}} = \underset{\nu_h \times 1}{\underset{\sim}{0}}$$

The hypothesis sum of squares, SS_h, for each hypothesis has the general form

(5)
$$SS_h = (C'\hat{\underset{\sim}{\mu}}_{\omega})'(C'(FD^{-1}F')C)^{-1}(C'\hat{\underset{\sim}{\mu}}_{\omega})$$

where

(6)
$$\hat{\underset{\sim}{\mu}}_{\omega} = (I-D^{-1}R(R'D^{-1}R)^{-1}R')\hat{\underset{\sim}{\mu}}_{\Omega}$$
$$= F\hat{\underset{\sim}{\mu}}_{\Omega}$$

$D = W'W$ and $\hat{\underset{\sim}{\mu}}_{\Omega} = (W'W)^{-1}W'\underset{\sim}{y} = D^{-1}W'\underset{\sim}{y}$. The sum of squares for error, under a fixed effect model, is

(7)
$$SS_e = (\underset{\sim}{y}-W\hat{\underset{\sim}{\mu}}_{\omega})'(\underset{\sim}{y}-W\hat{\underset{\sim}{\mu}}_{\omega})$$

The degrees of freedom associated with any SS_h is ν_h, the full row rank of C', and the degrees of freedom for SS_e is $\nu_e = N-p+s$ where N is the total number of observations, p is the full column rank of W, and s is the full row rank of R'.

If the variance-covariance matrix Σ for the data in Table 1 is not of the proper form for a univariate analysis, the correct analysis of the data is a multivariate analysis. The advantage of the multivariate analysis over the univariate analysis is that the analysis is valid whether or not Σ satisfies the compound symmetry assumption. In addition, univariate tests are easily obtained from some multivariate tests. To analyze split-plot repeated measures data using a multivariate model we again must have commensurable units over conditions, factor B; furthermore, the number of subjects within each level of A must be greater than or equal to the number of times the subjects are observed over the repeated measures factor.

The restricted full rank multivariate linear model is

$$\begin{aligned} \Omega: \quad & \underset{N \times p}{Y} = \underset{N \times q}{W} \; \underset{q \times p}{U} + \underset{N \times p}{E_o} \\ & \text{subject to the restrictions} \\ & \underset{r \times q}{R'} \; \underset{q \times p}{U} = \underset{r \times p}{0} \end{aligned} \tag{8}$$

where Y is a data matrix, W is a known full column rank design matrix, U is an unknown parameter matrix of population means, and E_o is the random error matrix. The matrix R' of full row rank r is the restriction matrix. Removing the restrictions from (8), the unrestricted multivariate linear model is represented by

$$\Omega: \; Y = WU + E_o \tag{9}$$

In either case, we assume that each row vector in Y follows a multivariate normal distribution and that

$$\begin{aligned} E(Y) &= WU \\ V(Y) &= I_N \otimes \Sigma \end{aligned} \tag{10}$$

To test hypotheses of the form

$$H: \; C'UA = 0 \tag{11}$$

under Ω, where $C'(\nu_h \times q)$ is a known matrix of real numbers of rank $\nu_h \leq q$ and $A(p \times t)$ is a known real matrix of rank $t \leq p$, hypotheses and error sum of squares and products matrices of the form

$$S_h = (C'\hat{U}_{\Omega}A)'(C'(FD^{-1}F')C)^{-1}(C'\hat{U}_{\Omega}A) \tag{12}$$
$$S_e = A'(SS_e)A$$

are constructed where

$$SS_e = (Y-W\hat{U}_{\Omega})'(Y-W\hat{U}_{\Omega})$$
$$\hat{U}_{\omega} = (I-D^{-1}R(R'D^{-1}R)^{-1}R')\hat{U}_{\Omega} \tag{13}$$
$$= F\hat{U}_{\Omega}$$
$$\hat{U}_{\Omega} = (W'W)^{-1}W'Y = D^{-1}W'Y$$

Letting $s = \min(t, \nu_h)$, multivariate hypotheses of the form given in (11) are directly tested using several multivariate criteria which are functions of the roots λ_1, λ_2, $\cdots$, and λ_s of the determinantal equation

$$|S_h-\lambda S_e| = 0 \tag{14}$$

A brief review of several multivariate criteria is contained in Timm and Carlson (1973). For the purposes of this paper, we shall use only Wilks' Λ-criterion

$$\Lambda = \frac{|S_e|}{|S_e+S_h|} = \prod_{i=1}^{s} (1+\lambda_i)^{-1} \tag{15}$$

The multivariate hypothesis is rejected if

$$\Lambda < U^{\alpha}(t, \nu_h, \nu_e) \tag{16}$$

where $\nu_e = N-q+r$ is the degrees of freedom for error. A general discussion of multivariate criteria as well as the construction of simultaneous confidence bounds for functions of the form $\psi = \underset{\sim}{c}'U\underset{\sim}{a}$ of the elements of U are discussed in Timm (1974). A general approach for investigating arbitrary functions of the elements of U, using a multivariate simultaneous test procedure, has been developed by Mudholkar, Davidson and Subbaiah (1973). Percentage points for the general U-distribution have been tabled by Wall (1967).

To test hypotheses under $\tilde{\Omega}$ or Ω, given a parameter matrix U of means, we merely have to construct matrices C' and A to represent the hypotheses. The general form of U for a multivariate analysis of a split-plot design is

(17)
$$\underset{I \times p}{U} = \begin{bmatrix} \mu_{11} & \mu_{12} & \cdots & \mu_{1p} \\ \mu_{21} & \mu_{22} & \cdots & \mu_{2p} \\ \vdots & \vdots & \cdots & \vdots \\ \mu_{I1} & \mu_{I2} & \cdots & \mu_{Ip} \end{bmatrix}$$

To analyze a split-plot design using the multivariate model, the unrestricted full rank multivariate model is employed. Multivariate tests of A, B, and AB, differences between groups, differences among profile conditions, and the interaction between groups and profile conditions, respectively, are represented by

(18)
$$A^*: \begin{bmatrix} \mu_{11} \\ \mu_{12} \\ \vdots \\ \mu_{1p} \end{bmatrix} = \begin{bmatrix} \mu_{21} \\ \mu_{22} \\ \vdots \\ \mu_{2p} \end{bmatrix} = \cdots = \begin{bmatrix} \mu_{I1} \\ \mu_{I2} \\ \vdots \\ \mu_{Ip} \end{bmatrix}$$

$$B^*: \begin{bmatrix} \mu_{11} \\ \mu_{21} \\ \vdots \\ \mu_{I1} \end{bmatrix} = \begin{bmatrix} \mu_{12} \\ \mu_{22} \\ \vdots \\ \mu_{I2} \end{bmatrix} = \cdots = \begin{bmatrix} \mu_{1p} \\ \mu_{2p} \\ \vdots \\ \mu_{Ip} \end{bmatrix}$$

$$(AB)^*: \begin{bmatrix} \mu_{11}-\mu_{12} \\ \mu_{12}-\mu_{13} \\ \vdots \\ \mu_{1(p-1)}-\mu_{1p} \end{bmatrix} = \begin{bmatrix} \mu_{21}-\mu_{22} \\ \mu_{22}-\mu_{23} \\ \vdots \\ \mu_{2(p-1)}-\mu_{2p} \end{bmatrix} = \cdots = \begin{bmatrix} \mu_{I1}-\mu_{I2} \\ \mu_{I2}-\mu_{I3} \\ \vdots \\ \mu_{I(p-1)}-\mu_{Ip} \end{bmatrix}$$

when profile data are analyzed. As shown by Timm and Carlson (1973), only the univariate test of AB can be recovered from the multivariate test (AB)*, provided the post matrix A, when stating the hypothesis in the form $C'UA = 0$, is appropriately normalized. This is not the case for the tests of A* and B*. The univariate tests of A and B cannot be recovered from the tests of A* and B*. Alternatively, if the test (AB)* is not significant or if we ignore the possibility of an interaction between A and B (groups and conditions), the tests of A and B are written as

$$\text{(19)} \quad \begin{aligned} A&: \sum_{j=1}^{p} \mu_{1j}/p = \sum_{j=1}^{p} \mu_{2j}/p = \cdots = \sum_{j=1}^{p} \mu_{Ij}/p \\ B&: \sum_{i=1}^{I} \mu_{i1}/I = \sum_{i=1}^{I} \mu_{i2}/I = \cdots = \sum_{i=1}^{I} \mu_{ip}/I \end{aligned}$$

if there are an equal number of subjects in each group. From these statements of A and B, the univariate tests of A and B are immediately obtained provided the post matrix A in the hypothesis, $C'UA = 0$, is appropriately normalized. That is, A must be normalized so that $A'A = I$. Several other multivariate tests are also testable using the multivariate representation for equal and unequal numbers of subjects within each level of A; however, these are discussed in detail by Timm and Carlson (1973) and will not be considered here.

To illustrate how we would test A*, B* and (AB)*, Kirk's data are reanalyzed. For Kirk's data,

$$U = \begin{pmatrix} \mu_{11} & \mu_{12} & \mu_{13} & \mu_{14} \\ \mu_{21} & \mu_{22} & \mu_{23} & \mu_{24} \end{pmatrix}$$

To test A*, the matrices

$$C'_{A*} = (1 \quad -1) \text{ and } A = I_2$$

are selected. To test B*, the matrices

$$C'_{B*} = I_2 \text{ and } A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & \vdots \\ 0 & 0 & -1 \end{pmatrix}$$

are used. Finally, (AB)* may be tested by using

$$C'_{(AB)*} = (1 \quad -1) \text{ and } A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

The MANOVA table for the analysis is shown in Table 7.

Table 7. Multivariate Profile Analysis I

Hypotheses	MSP $\doteq$ SSP/ν	DF	Λ	p-value
A*	$\begin{pmatrix} 8.00 & & & (\text{Sym}) \\ 4.00 & 2.00 & & \\ 6.00 & 3.00 & 4.50 & \\ -8.00 & -4.00 & -6.00 & 8.00 \end{pmatrix}$	1	0.137	0.1169
B*	$\begin{pmatrix} 3.25 & & (\text{Sym}) \\ 7.75 & 30.50 & \\ 11.75 & 28.50 & 42.50 \end{pmatrix}$	2	0.004	0.0002
(AB)*	$\begin{pmatrix} 2.00 & & (\text{Sym}) \\ -1.00 & 0.50 & \\ 7.00 & -3.50 & 24.50 \end{pmatrix}$	1	0.144	0.0371
Error				
A*	$\begin{pmatrix} 1.250 & & & \\ 0.667 & 0.667 & & \\ 0.583 & 0.333 & 0.500 & \\ 0.000 & -0.167 & 0.167 & 0.667 \end{pmatrix}$	6		
B*	$\begin{pmatrix} 0.583 & & (\text{Sym}) \\ -0.250 & 0.500 & \\ 0.083 & 0.167 & 0.833 \end{pmatrix}$	6		
(AB)*	$\begin{pmatrix} 0.583 & & (\text{Sym}) \\ -0.250 & 0.500 & \\ 0.083 & 0.167 & 0.833 \end{pmatrix}$	6		

Alternatively, testing A, B, and AB with the post matrix A normalized, the following matrices are employed

$$C'_A = (1 \quad -1) \text{ and } A = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{pmatrix}$$

$$C'_B = (\tfrac{1}{2} \quad \tfrac{1}{2}) \text{ and } A = \begin{pmatrix} 0.707107 & 0.408248 & 0.288675 \\ -0.707107 & 0.408248 & 0.288675 \\ 0.000000 & -0.816497 & 0.288675 \\ 0.000000 & 0.000000 & -0.866025 \end{pmatrix}$$

$$C'_{AB} = (1 \quad -1) \text{ and } A = \begin{pmatrix} 0.707107 & 0.408248 & 0.288675 \\ -0.707107 & 0.408248 & 0.288675 \\ 0.000000 & -0.816497 & 0.288675 \\ 0.000000 & 0.000000 & -0.866025 \end{pmatrix}$$

The MANOVA table for this analysis is displayed in Table 8.

Table 8. Multivariate Profile Analysis II

Hypotheses	MSP = SSP/ν	DF	Λ	p-value
A	3.125	1	0.250	0.2070
B	$\begin{pmatrix} 2.250 & & (Sym) \\ 10.825 & 52.083 & \\ 17.759 & 85.442 & 140.167 \end{pmatrix}$	1	0.027	0.0014
AB	$\begin{pmatrix} 1.000 & & (Sym) \\ 0.000 & 0.000 & \\ 4.287 & 0.000 & 18.375 \end{pmatrix}$	1	0.144	0.0371
Error				
A	1.563	6		
B	$\begin{pmatrix} 0.292 & & (Sym) \\ 0.024 & 0.264 & \\ 0.068 & 0.334 & 0.965 \end{pmatrix}$			
AB	$\begin{pmatrix} 0.292 & & (Sym) \\ 0.024 & 0.264 & \\ 0.068 & 0.334 & 0.965 \end{pmatrix}$			

From the entries in Table 8, the univariate F-ratios for testing A, B and AB are immediately obtained by averaging the diagonal elements in the MSP matrices, the corresponding diagonal elements in the error mean square and products matrices, and forming the ratio of these averages, Timm and Carlson (1973). That is,

$$F_A = \frac{3.125}{1.563} = 2.00 \sim F(1,6)$$

$$F_B = \frac{194.50/3}{1.521/3} = \frac{64.83}{0.51} = 127.88 \sim F(3,18)$$

$$F_{AB} = \frac{19.375/3}{1.521/3} = \frac{6.46}{0.51} = 12.74 \sim F(3,18)$$

The hypothesis and error degrees of freedom for each univariate F-ratio are obtained by multiplying the degrees of freedom for each multivariate test by the rank of the post matrix A corresponding to the test. Hence, for the ratio F_B the univariate degrees of freedom are

$$\nu_h = \nu_B R(A) = 1 \cdot 3 = 3$$

$$\nu_e = \nu_e R(A) = 6 \cdot 3 = 18$$

The others follow similarly.

To investigate trends in split-plot designs, Potthoff Roy's (1964) multivariate unrestricted full rank linear growth curve model

(20) $$\underset{N \times q}{Y_o} = \underset{N \times q}{W} \; \underset{q \times p}{B} \; \underset{p \times q}{P} + \underset{N \times q}{E_o}$$

is employed where

$$E(Y_o) = WBP$$

$$V(Y_o) = I_N \otimes \Sigma$$

The matrix Y_o is a data matrix, W is a known full rank design matrix, B is an unknown matrix of regression coefficients, P is a known matrix of rank $p \leq q$, E_o is the random error matrix, and the rows of Y_o are independently normally distributed. Implicit in the model is the assumption that each vector response variate can be expressed as a linear regression model of the form

$$\underset{\sim}{y}_i = P'\underset{\sim}{\beta}_i + \underset{\sim}{\epsilon}_i$$

where $y_i(q \times 1)$ is the observation vector for the i^{th} subject and β_i is a vector of unknown regression parameters.

To reduce model (20) to model (9), Potthoff and Roy suggested the following transformation of Y_o to Y,

(21) $$Y = Y_o G^{-1} P'(PG^{-1}P')^{-1}$$

where $G(q \times q)$ is any symmetric positive definite weight matrix either nonstochastic or independent of Y such that $PG^{-1}P'$ is of full rank. Since W has full column rank, an unbiased estimate of B, under the transformation, is

(22) $$\hat{B} = (W'W)^{-1}W'Y_o G^{-1}P'(PG^{-1}P')^{-1}$$

Under the transformation, each row of Y is normally distributed with variance-covariance matrix

$$\underset{p \times p}{\Sigma} = (PG^{-1}P')^{-1}PG^{-1}\Sigma_o G^{-1}P'(PG^{-1}P')^{-1}$$

however, the minimum variance unbiased estimator of B is

(23) $$\hat{B} = (W'W)^{-1}W'Y_o\Sigma_o^{-1}P'(P\Sigma_o P')^{-1}$$

This is unfortunate since Σ_o is usually unknown in practice.

Notice however, that if $p = q$, the transformation defined in (21) reduces to

(24) $$Y = Y_o P^{-1}$$

so that there is no need to choose G. Bock (1963b), developed a procedure for this case using orthogonal polynomials.

If $p < q$, however, the choice of G is important since it affects the power of tests and the widths of confidence bands. The variance of the estimator $\hat{B}$ increases as G^{-1} departs from Σ_o^{-1}. A simple choice of G is to set $G = I$. Then

(25) $$Y = Y_o P'(PP')^{-1}$$

Such a choice of G will certainly simplify one's calculations; however, it is not the best choice in terms of power since information is lost by reducing Y_o to Y unless G is set equal to Σ_o. This suggests obtaining an estimate of Σ_o based on the data. Khatri (1966) showed that the maximum likelihood estimator of B is given by

$$\hat{B} = (W'W)^{-1}W'Y_o\hat{\Sigma}_o^{-1}P'(P\hat{\Sigma}_o^{-1}P')^{-1} \tag{26}$$

where $\hat{\Sigma}_o$ is the maximum likelihood estimator of Σ_o defined by

$$\hat{\Sigma}_o = Y_o'(I-W(W'W)^{-1}W')Y_o/N$$

Alternatively, we could have used

$$S = Y_o'(I-W(W'W)^{-1}W')Y_o \text{ or } \hat{S} = S/(N-1)$$

instead of $\hat{\Sigma}_o$ since $\hat{B}$ does not change for these estimates.

To avoid the arbitrary selection of the matrix G under the Potthoff-Roy model, Rao (1965, 1966) using the method of covariance adjustment showed how additional information in the sample Y_o could be recovered by incorporating into the growth curve model (q-p) covariates. However, it was Grizzle and Allen (1969) that unified the previous approaches. They showed how Rao's results using a covariance model were identical to Khatri's result, and described a procedure for using stochastic weight matrices other than $\hat{\Sigma}_o^{-1}$ if it was desirable.

In summary, when $p < q$ the Potthoff-Roy reduction using $G = I$ is equivalent to not using covariates in the Rao-Khatri formulation. If the weight matrix $G = \hat{\Sigma}_o$ in the Potthoff-Roy transformation, this is equivalent to using q-p covariates in the Rao-Khatri reduction. When $p = q$, the Rao-Khatri procedure is not applicable since the Potthoff-Roy transformation does not depend on G. That is, the estimate for B obtained by Rao using (q-p) covariates, by

Khatri using the maximum likelihood procedure and by Potthoff and Roy weighting by $G^{-1} = \hat{\Sigma}_o^{-1}$ are identical. The variance-covariance matrix of $\hat{B}$ is given by

$$\text{Var}(\hat{B}) = (W'W)^{-1} \otimes \left(\frac{N-q-1}{N-g-q+p-1}\right) (P\Sigma_o^{-1}P')^{-1} \tag{27}$$

Rao (1967) and Williams (1967).

Hypotheses of interest under the growth curve model are usually stated in the form

$$H:\quad C'BA = 0 \tag{28}$$

where $C'(\nu_h \times g)$ is of full row rank ν_h and $A(p \times t)$ is of full column rank t. Using Potthoff and Roy's (1964) procedure for testing H, the data matrix Y_o is transformed to Y using the transformation

$$Y = Y_o G^{-1}P'(PG^{-1}P')^{-1}$$

and hypothesis and error sum of squares and products matrices become

$$\begin{aligned} S_h &= (C'\hat{B}A)'(C'(W'W)^{-1}C)^{-1}(C'\hat{B}A) \\ S_e &= A'Y'(I-W(W'W)^{-1}W')YA \end{aligned} \tag{29}$$

The hypothesis is rejected if

$$\Lambda < U^{\alpha}(t, \nu_h, \nu_e)$$

where $\nu_e = N-g$. However, Potthoff and Roy's formulation does not allow G to be stochastic unless it is independent of Y_o. That is, G must be chosen independent of the experimental data under investigation. Hence, we may not set $G = \hat{\Sigma}_o$.

Alternatively using the Rao-Khatri reduction, we may select $\hat{\Sigma}_o = G$; however, then

$$\begin{aligned} S_h &= (C'\hat{B}A)'(C'RC)^{-1}(C'\hat{B}A) \\ S_e &= A'(P\hat{\Sigma}_o^{-1}P')^{-1}A \end{aligned} \tag{30}$$

and

$$R = (W'W)^{-1}+(W'W)^{-1}W'Y_o(\hat{\Sigma}_o^{-1}-\hat{\Sigma}_o^{-1}P'(P\hat{\Sigma}_o^{-1}P')^{-1}P\hat{\Sigma}_o^{-1})Y_o'W(W'W)^{-1}$$

The hypothesis is rejected using this procedure if

$$\Lambda < U^{\alpha}(t, \nu_h, \nu_e)$$

where $\nu_e = N-g-q+p$. If $p = q$ both procedures give identical results.

To analyze Kirk's data using the growth curve model, the model is

$$\underset{8 \times 4}{Y_o} = \underset{8 \times 2}{W} \; \underset{2 \times 4}{B} \; \underset{4 \times 4}{P} + \underset{8 \times 4}{E_o},$$

If we assume that $p = q$ where

$$Y_o = \begin{bmatrix} 3 & 4 & 7 & 7 \\ 6 & 5 & 8 & 8 \\ 3 & 4 & 7 & 9 \\ 3 & 3 & 6 & 8 \\ 1 & 2 & 5 & 10 \\ 2 & 3 & 6 & 10 \\ 2 & 4 & 5 & 9 \\ 2 & 3 & 6 & 11 \end{bmatrix} \quad W = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} \beta_{10} & \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{20} & \beta_{21} & \beta_{22} & \beta_{23} \end{bmatrix}$$

Using a matrix of normalized orthogonal polynomials for the matrix P,

$$(31) \qquad P = \begin{bmatrix} .5 & .5 & .5 & .5 \\ -.67082 & -.22361 & .22361 & .67082 \\ .5 & -.5 & -.5 & .5 \\ -.22361 & .67082 & -.67082 & .22361 \end{bmatrix}$$

The functional form of the polynomial fit to the data using (31) is

$$E(Y_{oij}) = \hat{\beta}_{10}(\tfrac{1}{2}) + \hat{\beta}_{11}\left[\frac{2x-5}{\sqrt{20}}\right] + \hat{\beta}_{12}\left[\frac{x^2-5x+5}{2}\right] + \hat{\beta}_{13}\left[\frac{10x^3-75x^2+167x-105}{3\sqrt{20}}\right]$$

for x = 1, 2, 3, 4. Alternatively, an unnormalized matrix of orthogonal polynomials could have been used. Then

$$P = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -3 & -1 & 1 & 3 \\ 1 & -1 & -1 & 1 \\ -1 & 3 & -3 & 1 \end{pmatrix}$$

and

$$E(Y_{olj}) = \hat{\beta}_{10} + \hat{\beta}_{11}(2x-5) + \hat{\beta}_{12}(x^2-5x+5) + \hat{\beta}_{13}\left(\frac{10x^3-75x^2+167x-105}{3}\right)$$

for s = 1, 2, 3, 4. Finally, if a Vandermode matrix is used,

$$P = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 8 & 27 & 64 \end{pmatrix}$$

and

$$E(Y_{olj}) = \hat{\beta}_{10} + \hat{\beta}_{11}x + \hat{\beta}_{12}x^2 + \hat{\beta}_{13}x^3$$

for x = 1, 2, 3, 4. The reason for selecting (31) for P is that the transformation from Y_o to Y becomes

(32) $$Y = Y_oP^{-1} = Y_oP'$$

since P is an orthogonal matrix. Furthermore,

$$\hat{B} = (W'W)^{-1}W'Y_oP' = Y.P'$$

where Y. is the matrix of means. Notice however that if p < q, G = I and P is defined as in (31) with

$$\underset{q \times q}{P} = \begin{pmatrix} P_1 \\ -- \\ P_2 \end{pmatrix}$$

and $P_1(p \times q)$ the first p rows of P that the transformation from Y_o to Y is

$$Y = Y_o P_1'(PP_1')^{-1} = Y_o P_1'$$

so that

$$\hat{B} = Y_{.} P_1'$$

which is equivalent to eliminating q-p columns of $\hat{B}$ under the transformation given in (32) or ignoring the q-p covariates in the Rao-Khatri reduction. If $p < q$ under the Rao-Khatri formulation,

$$\hat{B} = Y_{.} \hat{\Sigma}_o^{-1} P_1 (P_1 \hat{\Sigma}_o^{-1} P_1')^{-1}$$

which is not the same as $\hat{B}$ using $G = I$.

In analyzing a split-plot design we first plot the means. For Kirk's data, the means are plotted in Figure 2.

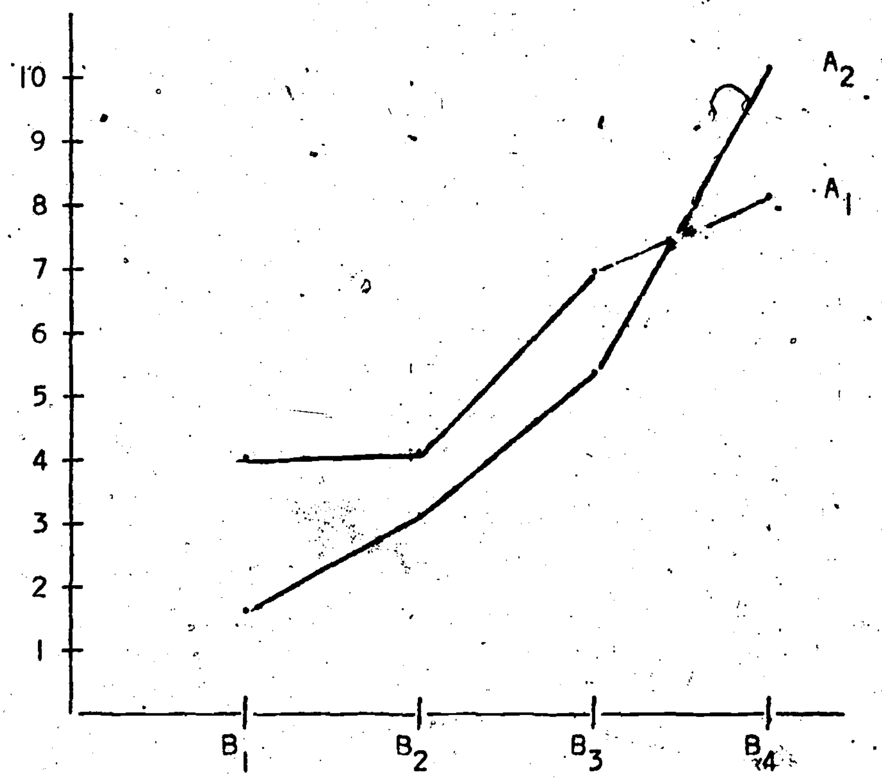

Figure 2. Means for Kirk's Data

From Figure 2, it appears that the trend for A_1 is cubic but that the trend for A_2 is quadratic. This distinction in growth curves cannot be made using model (20) since implicit in the model is the assumption that the expected value of each individual's response is from the same family (some (p-1) degree polymonial). This sometimes causes us to overfit some growth curves. Models which avoid this are discussed by Kleinbaum (1970, 1973b).

With the growth curves plotted, we can either determine whether the growth curves are quadratic rather than cubic and then test for parallelism with $p < q$ or we may test for parallelism with $p = q$ and then determine the degree of the polymonial required to describe the growth curves. For the first procedure, the matrices to test that the cubic term is zero are

$$C' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } A = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

These are used to test the hypothesis

$$H: \beta_{13} = \beta_{23} = 0$$

If this hypothesis is tenable, the parallelism hypothesis would be tested by setting

$$C' = (1 \quad -1),\ A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$(32) \qquad P = \begin{pmatrix} .5 & .5 & .5 & .5 \\ -.67082 & -.22361 & .22361 & .67082 \\ .5 & -.5 & -.5 & .5 \end{pmatrix}$$

with $p < q$ and $G = \hat{\Sigma}_o$ for the Rao-Khatri procedure or $G = I$ using the Potthoff-Roy technique. The hypothesis being tested is

(33) $$H_p: \begin{pmatrix} \beta_{11} \\ \beta_{12} \end{pmatrix} = \begin{pmatrix} \beta_{21} \\ \beta_{22} \end{pmatrix}$$

Alternatively, we may test for parallelism with $p = q$. The hypothesis is then

(34) $$H_p^*: \begin{pmatrix} \beta_{11} \\ \beta_{12} \\ \beta_{13} \end{pmatrix} = \begin{pmatrix} \beta_{21} \\ \beta_{22} \\ \beta_{23} \end{pmatrix}$$

and the matrices C' and A are defined by

$$C' = (1 \quad -1) \quad A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Using this latter procedure for Kirk's data,

$$\hat{B} = Y.P' = \begin{pmatrix} 11.375 & 3.522 & 0.375 & -1.062 \\ 10.125 & 6.093 & 1.625 & 0.168 \end{pmatrix}$$

$$\begin{aligned} S_h &= (C'\hat{B}A)'(C'(W'W)^{-1}C)^{-1}(C'\hat{B}A) \\ &= (C'Y.P'A)'(C'(W'W)^{-1}C)^{-1}(C'Y.P'A) \\ &= A^{*\prime}Y.'C(C'(W'W)^{-1}C)^{-1}C'Y.A^* \end{aligned}$$

where $A^* = P'A$ which satisfies the condition that $A^{*\prime}A^* = I$ as in profile analysis since P is orthonormal and

$$\begin{aligned} S_e &= A'PY_o'(I-W(W'W)^{-1}W')Y_oP'A \\ &= A^{*\prime}Y_o'(I-W(W'W)^{-1}W')Y_oA^* \end{aligned}$$

For Kirk's data,

$$S_h = \begin{bmatrix} 1.653 & & (\text{Sym}) \\ 0.804 & 0.391 & \\ 0.791 & 0.384 & 0.378 \end{bmatrix}, \quad S_e = \begin{bmatrix} 0.759 & & (\text{Sym}) \\ 0.203 & 0.234 & \\ 0.034 & -0.049 & 0.147 \end{bmatrix}$$

and Wilks' Λ-criteiron is $\Lambda = 0.144$. Comparing Λ with the $\alpha = 0.05$ critical value, $U^{\alpha}(3, 1, 6) = 0.1683$, the parallelism hypothesis is not tenable. The p-value for the test is $\alpha_p = 0.0371$. As expected, this result is identical to the multivariate test of interaction in profile analysis for the transformation given in (32). For this reason, the interaction test in profile analysis is often referred to as the test of parallelism.

To determine the degree of the polynomial required to describe the rejection of the parallelism hypothesis, univariate F-tests are constructed by dividing the diagonal elements of S_h and S_e by their degrees of freedom and forming univariate F-ratios. That is, beginning with the highest order term,

$$F_{\text{cubic}} = \frac{0.378/1}{0.147/6} = 15.43 \sim F(1, 6)$$

$$F_{\text{quadratic}} = \frac{0.391/1}{0.234/6} = 10.02 \sim F(1, 6)$$

$$F_{\text{linear}} = \frac{1.653/1}{.759/6} = 13.07 \sim F(1, 6)$$

Comparing each F-ratio with $F^{\alpha^*}(1, 6) = 11.09$, where $\alpha^* = \alpha/3 = 0.05/3 = 0.0167$ to control the error rate at a nominal level less than or equal to α, we begin by testing the higher order terms first and we stop testing when the first significant F-ratio is reached. For Kirk's data, a cubic trend best describes nonparallelism. This procedure is seen to be identical with the univariate analysis. If $p < q$, this is not the case.

If the parallelism hypothesis is tenable, we would next test to see whether the growth curves are coincident. Of course, we could test this without first testing for parallelism. To test for coincidence with $p < q$ (the cubic term being zero, say), the hypothesis is

$$(35) \qquad H_c: \begin{pmatrix} \beta_{10} \\ \beta_{11} \\ \beta_{12} \end{pmatrix} = \begin{pmatrix} \beta_{20} \\ \beta_{21} \\ \beta_{22} \end{pmatrix}$$

with

$$C' = (1 \quad -1),\ A = I_3$$

P defined in (32), and $G = I$ or $G = \hat{\Sigma}_o$ depending on whether we use the Potthoff-Roy model or the Rao-Khatri model. With $p=q$, the test for coincidence is

$$(36) \qquad H_c^*: \begin{pmatrix} \beta_{10} \\ \beta_{11} \\ \beta_{12} \\ \beta_{13} \end{pmatrix} = \begin{pmatrix} \beta_{20} \\ \beta_{21} \\ \beta_{22} \\ \beta_{23} \end{pmatrix}$$

Then

$$C' = (1 \quad -1),\ A = I_4$$

and P in (31) are used. When $p = q$, this test is identical to testing the hypothesis A* in profile analysis.

Another test which may be of interest when $p = q$ is the growth hypothesis

$$(37) \qquad H_G^*: \begin{pmatrix} \beta_{11} \\ \beta_{21} \end{pmatrix} = \begin{pmatrix} \beta_{12} \\ \beta_{22} \end{pmatrix} = \begin{pmatrix} \beta_{13} \\ \beta_{23} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This is identical to the test of B* in profile analysis. This is not the case if $p < q$.

If the parallelism hypothesis is tenable, but the coincidence test is rejected, we may still investigate trends over B with the trend hypothesis

(38) $$H_T: \sum_{i=1}^{2} \beta_{i1} = \sum_{i=1}^{2} \beta_{i2} = \sum_{i=1}^{2} \beta_{i3} = 0$$

if p = q. For this test,

$$C' = (1 \quad 1) \text{ and } A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This is equivalent to the test of B studied in profile analysis. To see this using Kirk's data, even though the parallelism hypothesis is not tenable,

$$S_h = \begin{pmatrix} 23.112 & & (\text{Sym}) \\ 4.808 & 1.000 & \\ -2.150 & -0.447 & 0.200 \end{pmatrix} \text{ and } S_e = \begin{pmatrix} 0.759 & & (\text{Sym}) \\ 0.203 & 0.234 & \\ 0.034 & -0.049 & 0.147 \end{pmatrix}$$

so that $\Lambda = 0.027$ which is compared to $U^{0.05}_{(3,\ 1,\ 6)} = 0.1683$. As claimed this is the result reported for testing B in Table 8. To investigate trend following the test of H_T, we proceed as we did when testing for parallelism when p = q. That is,

$$F_{\text{cubic}} = \frac{0.200/1}{0.147/6} = 8.16 \sim F(1,\ 6)$$

$$F_{\text{quadratic}} = \frac{1.000/1}{0.234/6} = 25.64 \sim F(1,\ 6)$$

$$F_{\text{linear}} = \frac{23.112/1}{0.759/6} = 182.71 \sim F(1,\ 6)$$

As expected, these results agree with those given in Table 3.

If $p < q$, the test of H_T would be written as

(39) $$H_T: \sum_{i=1}^{2} \beta_{i1} = \sum_{i=1}^{2} \beta_{i2} = 0$$

if the cubic term is zero.

Given that both the coincidence and parallelism hypotheses are tenable, we would analyze the data for trend over B with either the Potthoff-Roy or Rao-Khatri models by treating the data as a single group. The advantage of this procedure is that the degrees of freedom for error are increased since the variation attributed to the group by conditions variation is pooled with error.

3. Multivariate Profile Analysis of Split-Split Plot Designs

The problems inherent in the analysis of a split-plot design become more complex in a split-split plot design. The classical univariate mixed model for the simplest type of split-split plot design is

$$y_{ijkm} = \mu+\alpha_i+\beta_k+\gamma_m+(\alpha\beta)_{ik}+(\alpha\gamma)_{im}+(\beta\gamma)_{km}+(\alpha\beta\gamma)_{ikm} \tag{40}$$
$$+s_{(i)j}+(\beta s)_{(i)jk}+(\gamma s)_{(i)jm}+\varepsilon_{(i)jkm}$$
$$i=1, \ldots, I;\ j=1, \ldots, J;\ k=1, \ldots, K;\ m=1, \ldots, M$$

where $s_{(i)j} \sim IN(0, \rho\sigma^2)$, $(\beta s)_{(i)jk} \sim IN(0, \rho\sigma^2)$, $(\gamma s)_{(i)jm} \sim IN(0, \rho\sigma^2)$, $\varepsilon_{(i)jkm} \sim IN(0, (1-\rho)\sigma^2)$ and $\varepsilon_{(i)jkm}$, $(\gamma s)_{(i)jm}$, $(\beta s)_{(i)jk}$, $s_{(i)j}$ are jointly independent. Thus, the variance-covariance matrix satisfies the compound symmetry assumption. Assuming a 3^2 factorial design within subjects, the layout for the design specified in (40) is given in Figure 3. The s_i in Figure 3 indicate that subjects within each level of A are observed repeatedly over all levels of B and C so that the units of measurement are commensurable over B and C.

	B_1			B_2			B_3		
	C_1	C_2	C_3	C_1	C_2	C_3	C_1	C_2	C_3
A_1	s_1	s_1	s_1	s_1	s_1	s_1	s_1	s_1	s_1
A_2	s_2	s_2	s_2	s_2	s_2	s_2	s_2	s_2	s_2

Figure 3. Simple Split-Plot Design

Using the artificial data in Table 9 and the restricted full rank linear model

$$y_{ijkm} = \mu_{ijkm} + \varepsilon_{(i)jkm}$$

(41) subject to the $I(J-1)(K-1)(M-1)$ linearly independent restrictions

$$\mu_{ijkm}-\mu_{ij'km}-\mu_{ijk'm}+\mu_{ij'k'm}-\mu_{ijkm'}+\mu_{ij'km'}+\mu_{ijk'm'}-\mu_{ij'k'm'} = 0$$

the ANOVA analysis for the data is shown in Table 10.

Table 9. Data for Split-Split Plot Design

		B_1			B_2			B_3		
		C_1	C_2	C_3	C_1	C_2	C_3	C_1	C_2	C_3
A_1	s_1	20	21	21	32	42	37	32	32	32
	s_2	67	48	29	43	56	48	39	40	41
	s_3	37	31	25	27	28	30	31	33	34
	s_4	42	40	38	37	36	28	19	27	35
	s_5	57	45	32	27	21	25	30	29	29
	s_6	39	39	38	46	54	43	31	29	28
	s_7	43	32	20	33	46	44	42	37	31
	s_8	35	34	34	39	43	39	35	39	42
	s_9	41	32	23	37	51	39	27	28	30
	s_{10}	39	32	24	30	35	31	26	29	32
A_2	s'_1	47	36	25	31	36	29	21	24	27
	s'_2	53	43	32	40	48	47	46	50	54
	s'_3	38	35	33	38	42	45	48	48	49
	s'_4	60	51	41	54	67	60	53	52	50
	s'_5	37	36	35	40	45	40	34	40	46
	s'_6	59	48	37	45	52	44	36	44	52
	s'_7	67	50	33	47	61	46	31	41	50
	s'_8	43	35	27	32	36	35	33	33	32
	s'_9	64	59	53	58	62	51	40	42	43
	s'_{10}	41	38	34	41	47	42	37	41	46

Table 10. Univariate Profile Analysis

Hypothesis	SS	DF	MS	F	P-value
1. Constant	275968.66	1	275968.66	$(\frac{1}{3}) = 775.08$	<.0001
2. A	3042.22	1	3042.22	$(\frac{2}{3}) = 8.54$	0.0091
3. S(A)	6408.90	18	356.05		
4. B	634.84	2	317.42	$(\frac{4}{6}) = 3.29$	0.0487
5. AB	18.71	2	9.36	$(\frac{5}{6}) = 0.10$	0.9051
6. SB(A)	3489.48	36	96.93		
7. C	427.81	2	213.91	$(\frac{7}{9}) = 14.95$	<.0001
8. AC	6.21	2	3.11	$(\frac{8}{9}) = 0.22$	0.8036
9. SC(A)	515.16	36	14.31		
10. BC	2440.89	4	610.22	$(\frac{10}{12}) = 27.64$	<.0001
11. ABC	67.36	4	16.84	$(\frac{11}{12}) = 0.76$	0.5547
12. Error	1589.76	72	22.08		
Total	294610.00	180			

Using the population means in Table 11 for the j^{th} subject within the i^{th} level of A, the hypotheses being tested in terms of the full rank model parameters are summarized in Table 12.

Table 11. Population Means

B_1			B_2			B_3		
C_1	C_2	C_3	C_1	C_2	C_3	C_1	C_2	C_3
μ_{ij11}	μ_{ij12}	μ_{ij13}	μ_{ij21}	μ_{ij22}	μ_{ij23}	μ_{ij31}	μ_{ij32}	μ_{ij33}

Table 12. Univariate Profile Analysis Hypotheses

Source	Hypotheses	DF
Constant	$\mu_{....} = 0$	1
A	all $\mu_{i...}$'s are equal	I-1
S(A)	$\mu_{11..} = \mu_{12..} = \cdots = \mu_{1J..}$ $\vdots \quad \vdots \quad \cdots \quad \vdots$ $\mu_{I1..} = \mu_{I2..} = \cdots = \mu_{IJ..}$	I(J-1)
B	all $\mu_{..k.}$'s are equal	K-1
AB	$\mu_{i.k.}-\mu_{i'.k.}-\mu_{i.k'.}+\mu_{i'.k'.} = 0$	(I-1)(J-1)
SB(A)	$\mu_{1jk.}-\mu_{1j'k.}-\mu_{1jk'.}+\mu_{1j'k'.} = 0$ $\vdots \quad \vdots \quad \vdots \quad \vdots$ $\mu_{Ijk.}-\mu_{Ij'k.}-\mu_{Ijk'.}+\mu_{Ij'k'.} = 0$	I(J-1)(K-1)
C	all $\mu_{...m}$'s are equal	(M-1)
AC	$\mu_{i..m}-\mu_{i'..m}-\mu_{i..m'}+\mu_{i'..m'} = 0$	(I-1)(M-1)
SC(A)	$\mu_{1j.m}-\mu_{1j'.m}-\mu_{1j.m'}+\mu_{1j'.m'} = 0$ $\vdots \quad \vdots \quad \vdots \quad \vdots$ $\mu_{Ij.m}-\mu_{Ij'.m}-\mu_{Ij.m'}+\mu_{Ij'.m'} = 0$	I(J-1)(M-1)
BC	$\mu_{..km}-\mu_{..k'm}-\mu_{..km'}+\mu_{..k'm'} = 0$	(K-1)(M-1)
ABC	$\mu_{i.jm}-\mu_{i'.jm}-\mu_{i.j'm}+\mu_{i'.j'm}$ $\quad -\mu_{i.jm'}+\mu_{i'.jm'}+\mu_{i.j'm'}-\mu_{i'.j'm'} = 0$	(I-1)(K-1)(M-1)
Error	(Same as the restrictions in (41))	
Total		IJKM

To analyze the data in Table 9 using the unrestricted full rank multivariate linear model, the means in Table 13 are used to construct U and the design matrix W is

$$W = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \tag{42}$$

Table 13. Multivariate Means

	B_1			B_2			B_3		
	C_1	C_2	C_3	C_1	C_2	C_3	C_1	C_2	C_3
A_1	μ_{11}	μ_{12}	μ_{13}	μ_{14}	μ_{15}	μ_{16}	μ_{17}	μ_{18}	μ_{19}
A_2	μ_{21}	μ_{22}	μ_{23}	μ_{24}	μ_{25}	μ_{26}	μ_{27}	μ_{28}	μ_{29}

The first hypothesis of interest using the multivariate model is to see whether there is an interaction between A and the levels of B, C and BC which we shall now call the test of parallelism. Following the formulation for testing interaction (parallelism) for a split-plot design, the matrices needed to test for parallelism, with the hypothesis stated in the form $C'UA = 0$, are

$$(43) \qquad C' = (1\ -1) \text{ and } A = \begin{array}{c} \left[\begin{array}{cccccccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ -1 & 1 & 1 & 0 & -1 & 0 & 1 & 0 \\ -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \\ \underbrace{\qquad\quad}_{B}\ \ \underbrace{\qquad\quad}_{C}\ \ \underbrace{\qquad\qquad\quad}_{BC}\qquad \end{array}$$

when

$$(44) \qquad U = \begin{bmatrix} \mu_{11} & \mu_{12} & \mu_{13} & \mu_{14} & \mu_{15} & \mu_{16} & \mu_{17} & \mu_{18} & \mu_{19} \\ \mu_{21} & \mu_{22} & \mu_{23} & \mu_{24} & \mu_{25} & \mu_{26} & \mu_{27} & \mu_{28} & \mu_{29} \end{bmatrix}$$

The first two columns of the post matrix A are formed to evaluate AB, the next two are used to investigate AC, and the last four, constructed from the first two by taking Hadamard vector products, are used to test ABC. Normalizing the post matrix A so that $A'A = I$ and separating out the submatrices used to test AB, AC, and ABC, we average the diagonal elements of the sub-hypothesis MSP and error component matrices, as we did for the split-plot design, to obtain the univariate F-ratios. If the design on B and C is a 2^2 rather than a 3^2 design, the diagonal elements of the MSP matrix contain the univariate mean squares for testing the univariate hypotheses AB, AC, and ABC.

To test BC, given that the parallelism hypothesis is tenable, the matrices

(45) $$C' = (1 \quad 1) \text{ and } A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

are used. The post matrix A is constructed by arranging the elements of U in table form, Table 14, and forming four linearly independent contrasts such that

(46) $$\eta_{ij}-\eta_{i'j}-\eta_{ij'}+\eta_{i'j'} = 0$$

Table 14. Rearranged Means for the Split-Split Plot Design

	C_1	C_2	C_3
B_1	$\mu_{11} = \eta_{11}$	$\mu_{12} = \eta_{12}$	$\mu_{13} = \eta_{13}$
B_2	$\mu_{14} = \eta_{21}$	$\mu_{15} = \eta_{22}$	$\mu_{16} = \eta_{23}$
B_3	$\mu_{17} = \eta_{31}$	$\mu_{18} = \eta_{32}$	$\mu_{19} = \eta_{33}$

	C_1	C_2	C_3
B_1	$\mu_{21} = \eta_{11}$	$\mu_{22} = \eta_{12}$	$\mu_{23} = \eta_{13}$
B_2	$\mu_{24} = \eta_{21}$	$\mu_{25} = \eta_{12}$	$\mu_{26} = \eta_{23}$
B_3	$\mu_{27} = \eta_{31}$	$\mu_{28} = \eta_{32}$	$\mu_{29} = \eta_{33}$

Normalizing A and averaging the diagonal elements of the hypothesis and error mean squares and products matrices, the univariate F-ratio for testing BC is immediately obtained. If the parallelism hypothesis is not tenable, we may test BC with (BC)* by using $C' = I_2$ and the post matrix A defined in (45). The univariate test of BC is not obtained from testing (BC)*.

To test the main effect hypotheses A, B and C, under parallelism and no BC interaction, we use the following matrices for C' and the post matrices A when the hypotheses are expressed in the form $C'UA = 0$:

$$(A):\quad C' = (1 \quad -1) \quad A = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$(B):\quad C' = (1 \quad 1) \quad A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ -1 & -1 \\ -1 & -1 \\ -1 & -1 \end{pmatrix}$$

(47)

$$C' = (1 \quad 1) \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \\ 1 & 0 \\ 0 & 1 \\ -1 & -1 \\ 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}$$

Normalizing the post matrices A in (47), univariate tests are easily obtained. Tests of A, B, and C which do not require parallelism are denoted by A*, B*, and C*. The matrices defined below are used to test these hypotheses.

$$(A^*): \quad C' = (1 \ -1) \quad A = I_9$$

$$(B^*): \quad C' = I_2 \quad A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ -1 & -1 \\ -1 & -1 \\ -1 & -1 \end{pmatrix}$$

(48)

$$(C^*):\quad C' = I_2 \qquad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \\ 1 & 0 \\ 0 & 1 \\ -1 & -1 \\ 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}$$

To write each of the multivariate hypotheses in terms of the elements of U, given in (44) for the example, we merely have to substitute the hypothesis test matrix C' and the post matrix A into the general expression for C'UA = 0 for each hypothesis. To test each of the preceding hypotheses, the expressions for S_h and S_e given in (12) are evaluated using the data in Table 9. To show the correspondence between the multivariate analysis in Table 15 and the univariate analysis in Table 10, the mean square and products matrices are displayed; the post matrix A for each hypothesis without an asterisk (*) has been normalized so that A'A = I.

Averaging the diagonal elements of the hypothesis test matrices of AB, AC and ABC within Paral, BC, C, B and A in Table 15 and the diagonal elements of the corresponding error matrices, univariate split-split plot F-ratios are immediately constructed. To illustrate, we test ABC and C:

$$F_{ABC} = \frac{(5.513+2.604+11.704+47.535)/4}{(20.357+31.497+8.778+27.700)/4} = \frac{16.48}{22.08} = .76$$

$$F_C = \frac{(261.075+166.736)/2}{(23.475+\ 5.141)/2} = \frac{213.91}{14.31} = 14.95$$

Table 15. Multivariate Profile Analysis of a Split-Split Plot Design

Hyp.	DF	MSP	Λ
Paral	1	AB AC ABC $\begin{pmatrix} 0.033 & & & & & & & \text{(Sym)} \\ 0.789 & 18.678 \\ 0.367 & 8.679 & 4.033 \\ 0.269 & 6.378 & 2.964 & 2.178 \\ -0.429 & -10.147 & -4.715 & -3.465 & 5.513 \\ -0.295 & -6.974 & -3.241 & -2.381 & 3.789 & 2.604 \\ -0.625 & -14.785 & -6.871 & -5.049 & 8.032 & 5.521 & 11.704 \\ -1.529 & -29.797 & -13.846 & -10.174 & 16.188 & 11.126 & 23.587 & 47.535 \end{pmatrix}$	0.809
BC	1	$\begin{pmatrix} 1872.113 & & & \text{(Sym)} \\ 13.965 & 0.104 \\ -712.198 & -5.313 & 270.937 \\ 746.587 & 5.569 & -284.020 & 297.735 \end{pmatrix}$	0.225
C	1	$\begin{pmatrix} 261.075 & \text{(Sym)} \\ 208.640 & 166.736 \end{pmatrix}$	0.325
B	1	$\begin{pmatrix} 154.133 & \text{(Sym)} \\ 272.201 & 480.711 \end{pmatrix}$	0.676
A	1	3042.22	0.678
(BC)*	2	$\begin{pmatrix} 3870.251 & & & \text{(Sym)} \\ 1956.651 & 989.300 \\ 723.750 & 364.200 & 166.500 \\ 1508.051 & 762.200 & 285.900 & 588.100 \end{pmatrix}$	0.209
C*	2	$\begin{pmatrix} 783.23 & \text{(Sym)} \\ 933.67 & 1113.03 \end{pmatrix}$	0.316
B*	2	$\begin{pmatrix} 506.50 & \text{(Sym)} \\ 980.40 & 1944.40 \end{pmatrix}$	0.670
A*	1	(deleted, lack of space)	0.533
Error Paral	18	AB AC ABC $\begin{pmatrix} 96.313 & & & & & & & \text{(Sym)} \\ 25.606 & 97.551 \\ 20.619 & 1.749 & 7.185 \\ 9.998 & 24.640 & 6.836 & 20.801 \\ -9.445 & 10.182 & 0.982 & 16.135 & 20.357 \\ 14.754 & -7.007 & 10.144 & 18.444 & 18.335 & 31.497 \\ 22.132 & 5.946 & 2.214 & 0.538 & -0.256 & 2.997 & 8.778 \\ 20.531 & 25.214 & -0.870 & 6.548 & 9.002 & 5.106 & 12.761 & 27.700 \end{pmatrix}$	
BC	18	$\begin{pmatrix} 44.424 & & & \text{(Sym)} \\ -0.314 & 0.099 \\ -9.057 & 0.002 & 34.188 \\ 6.008 & -0.025 & -2.217 & 9.622 \end{pmatrix}$	

Hyp.	DF	MSP	Λ
C	18	$\begin{pmatrix} 23.475 & (Sym) \\ 2.205 & 5.141 \end{pmatrix}$	
B	18	$\begin{pmatrix} 119.417 & (Sym) \\ -12.267 & 5.141 \end{pmatrix}$	
A	18	356.05	
(BC)*	18	$\begin{pmatrix} 117.694 & & & (Sym) \\ 87.761 & 43.633 & & \\ 57.472 & 28.200 & 115.611 & \\ 46.217 & 22.922 & 136.775 & 53.544 \end{pmatrix}$	
C*	18	$\begin{pmatrix} 140.850 & (Sym) \\ 81.883 & 628.250 \end{pmatrix}$	
B*	18	$\begin{pmatrix} 716.500 & (Sym) \\ 294.511 & 450.400 \end{pmatrix}$	
A*	18	(deleted, lack of space	

p-values

Paral	0.9392	B	0.0356	C*	0.0005
BC	<.0001	A	0.0091	B*	0.1357
C	<.0001	(BC)*	0.0012	A*	0.5114

Using the growth curve model in (20), a trend analysis of the data in Table 9 is easily carried out once the post matrix P is constructed. Unlike the split-plot example, trends over the within subjects dimension, factors B and C, are not over all nine points but within levels of B and C. That is, we have only three time points and not nine. The unnormalized orthogonal polynomials for three time points are

		t_1	t_2	t_3	Sum of Squares
Constant	$\underset{\sim}{C}' =$	1	1	1	3
Linear	$\underset{\sim}{L}' =$	-1	0	1	2
Quadratic	$\underset{\sim}{Q}' =$	1	-2	1	6

To extend these over nine points, defined by a 3^2 factorial design, we form the following Kronecker products

	B_1			B_2			B_3		
	C_1	C_2	C_3	C_1	C_2	C_3	C_1	C_2	C_3
$\underset{\sim}{C}' \otimes \underset{\sim}{C}' =$	1	1	1	1	1	1	1	1	1
$\underset{\sim}{L}' \otimes \underset{\sim}{C}' =$	-1	-1	-1	0	0	0	1	1	1
$\underset{\sim}{Q}' \otimes \underset{\sim}{C}' =$	1	1	1	-2	-2	-2	1	1	1
$\underset{\sim}{C}' \otimes \underset{\sim}{L}' =$	-1	0	1	-1	0	1	-1	0	1
$\underset{\sim}{C}' \otimes \underset{\sim}{Q}' =$	1	-2	1	1	-2	1	1	-2	1
$\underset{\sim}{L}' \otimes \underset{\sim}{L}' =$	1	0	-1	0	0	0	-1	0	1
$\underset{\sim}{L}' \otimes \underset{\sim}{Q}' =$	-1	2	-1	0	0	0	1	-2	1
$\underset{\sim}{Q}' \otimes \underset{\sim}{L}' =$	-1	0	1	2	0	-2	-1	0	1
$\underset{\sim}{Q}' \otimes \underset{\sim}{Q}' =$	1	-2	1	-2	4	-2	1	-2	1

Using normalized polynomials instead of unnormalized polynomials, the matrix P for the trend analysis is defined by

$$P = \begin{bmatrix}
.333333 & .333333 & .333333 & .333333 & .333333 & .333333 & .333333 & .333333 & .333333 \\
-.408248 & -.408248 & -.408248 & .000000 & .000000 & .000000 & .408248 & .408248 & .408248 \\
.235702 & .235702 & .235702 & -.471405 & -.471405 & -.471405 & .235702 & .235702 & .235702 \\
-.408248 & .000000 & .408248 & -.408248 & .000000 & .408248 & -.408248 & .000000 & .408248 \\
.235702 & .471405 & .235702 & .235702 & -.471405 & .235702 & .235702 & -.471405 & .235702 \\
.500000 & .000000 & -.500000 & .000000 & .000000 & .000000 & -.500000 & .000000 & .500000 \\
-.288675 & .577350 & -.288675 & .000000 & .000000 & .000000 & .288675 & -.577350 & .288675 \\
-.288675 & .000000 & .288675 & .577350 & .000000 & -.577350 & -.288675 & .000000 & .288675 \\
.166667 & -.333333 & .166667 & -.333333 & .666667 & -.333333 & .166667 & -.333333 & .166668
\end{bmatrix}$$

Analyzing trends using this procedure has already been discussed by Bock (1963b) and will not be considered here. For this type of analysis p is always equal to q unless the model over the within dimension is additive. For an additive model we may either select $G = I$, using the Potthoff-Roy model, or $G = \hat{\Sigma}_o$, using the Rao-Khatri formulation.

4. Growth Curve Analysis and Profile Analysis of Multivariate Repeated Measurements

The repeated measures type designs discussed so far in this paper were such that the researcher obtained commensurable measurements on each subject over time or over several experimental conditions. In many experimental situations, especially longitudinal studies, we often collect data on several variates and observe a subject on each of the variates over time. For profile analysis the time points are unordered experimental conditions. Designs with multivariate observations on p variates observed over q time points or q conditions are called multivariate or multi-response repeated measures designs since the multivariate observations are not commensurable at each time point but are commensurable over time or conditions, a variable at a time, Figure 4.

Treatment	Subject	t_1	t_2	...	t_q
A_1	s_1	$\underset{\sim}{y}_{1j1} = \begin{pmatrix} y^{(1)}_{1j1} \\ y^{(1)}_{1j2} \\ \vdots \\ y^{(1)}_{1jp} \end{pmatrix}$	$\underset{\sim}{y}_{1j2} = \begin{pmatrix} y^{(2)}_{1j1} \\ y^{(2)}_{1j2} \\ \vdots \\ y^{(2)}_{1jp} \end{pmatrix}$	...	$\underset{\sim}{y}_{1jq} = \begin{pmatrix} y^{(q)}_{1j1} \\ y^{(q)}_{1j2} \\ \vdots \\ y^{(q)}_{1jp} \end{pmatrix}$

Figure 4. p-variate Observations over q Conditions

Since each of the p-variates are observed over q time points (or conditions), it is convenient to rearrange the data in Figure 4 by variates for a multivariate repeated measures analysis so that each variate is observed over q periods, Figure 5. The data matrix Y for the analysis is of order $N \times pq$ where the first q columns correspond to variable one, the next q to variable 2, the next q to variable 3 and so on up to the p^{th} variable. Alternatively, using the data as arranged in Figure 4 a multivariate mixed model analysis of variance procedure may be used to analyze multi-response repeated measures data. This would be done by simply extending the univariate sum of squares in Table 2 to sum of squares and products matrices and calculating multivariate criteria to test hypotheses. However, for such an analysis we must not only assume a restrictive structure on the valrance-covariance matrix associated with each variable over q time points but that the structure on each variance-covariance matrix between variables across time points is constant. This is even more restrictive than the univariate assumptions and for this reason is not usually recommended. Instead, a multivariate approach should be used.

Treatment	Subject	1	2	...	p
A_1	s_1	$\underset{\sim}{y}_{1j1} = \begin{bmatrix} y^{(1)}_{1j1} \\ y^{(2)}_{1j1} \\ \vdots \\ y^{(q)}_{1j1} \end{bmatrix}$	$\underset{\sim}{y}_{1j2} = \begin{bmatrix} y^{(1)}_{1j2} \\ y^{(2)}_{1j2} \\ \vdots \\ y^{(q)}_{1j2} \end{bmatrix}$	...	$\underset{\sim}{y}_{1jp} = \begin{bmatrix} y^{(1)}_{1jp} \\ y^{(2)}_{1jp} \\ \vdots \\ y^{(q)}_{1jp} \end{bmatrix}$

Figure 5. Data Layout for Trend Analysis of Multivariate Repeated Measures Design

The data in Figure 5 may be either trend or profile data. Although the analysis of profile data and trend data are very similar, as seen for example in the analysis of a split-plot design, we shall consider each analysis separately. For a growth curve analysis of the data in Figure 5, the model given in (20) is employed. To use the model, the number of subjects within each treatment must be greater than or equal to the total number of measurements on each subject. For simplicity, suppose that three measures are recorded for each subject at q time points so that associated with each subject are 3q measurements all correlated with unknown variance-covariance matrix Σ_o. Letting the i^{th} growth curve for each of the three multivariate responses be represented by

$$
\begin{array}{lll}
 & \beta_{i0}+\beta_{i1}t+\dots+\beta_{i,p_1-1}t^{p_1-1} & p_1 \le q \\
(49) & \theta_{i0}+\theta_{i1}t+\dots+\theta_{i,p_2-1}t^{p_2-1} & p_2 \le q \\
 & \xi_{i0}+\xi_{i1}t+\dots+\xi_{i,p_3-1}t^{p_3-1} & p_3 \le q
\end{array}
$$

the matrices B and P are defined as follows:

$$
\underset{g \times (p_1+p_2+p_3)}{B} = \begin{pmatrix}
\beta_{10} & \beta_{11} & \dots & \beta_{1,p_1-1} & \theta_{10} & \theta_{11} & \dots & \theta_{1,p_2-1} & \xi_{10} & \xi_{11} & \dots & \xi_{1,p_3-1} \\
\beta_{20} & \beta_{21} & \dots & \beta_{2,p_1-1} & \theta_{20} & \theta_{21} & \dots & \theta_{2,p_2-1} & \xi_{20} & \xi_{21} & \dots & \xi_{2,p_3-1} \\
\vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\
\beta_{g0} & \beta_{g1} & \dots & \beta_{g,p_1-1} & \theta_{g0} & \theta_{g1} & \dots & \theta_{g,p_2-1} & \xi_{g0} & \xi_{g1} & \dots & \xi_{g,p_3-1}
\end{pmatrix}
$$

(50)

$$
\underset{(p_1+p_2+p_3) \times 3q}{P} = \begin{pmatrix}
1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\
t_1 & t_2 & \dots & t_q & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\
t_1^{p_1-1} & t_2^{p_1-1} & \dots & t_q^{p_1-1} & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 \\
0 & 0 & \dots & 0 & t_1 & t_2 & \dots & t_q & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\
0 & 0 & \dots & 0 & t_1^{p_2-1} & t_2^{p_2-1} & \dots & t_q^{p_2-1} & 0 & 0 & \dots & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & t_1 & t_2 & \dots & t_q \\
\vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & t_1^{p_3-1} & t_2^{p_3-1} & \dots & t_q^{p_3-1}
\end{pmatrix}
$$

The design matrix W is

$$\underset{N \times g}{W} = \begin{pmatrix} \underset{\sim}{1}_{J_1} & 0 & \cdots & 0 \\ 0 & \underset{\sim}{1}_{J_2} & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \underset{\sim}{1}_{J_g} \end{pmatrix}$$

where $\underset{\sim}{1}_{J_i}$ is a vector of J_i unities for J_i subjects within each of g groups where $J_i \geq pg$. The matrix P given above has been represented as a Vandermode matrix; alternatively, we could have used unnormalized or normalized orthogonal polynomials.

Tests commonly investigated with multi-response growth curve data are similar to uni-response data; however, we are interested in analyzing trends for several different variables simultaneously where the degree of the polynomial which "best" describes one variate may be different for another variate. We are, however, still restricted to fitting a polynomial of the same degree to all treatment groups.

To illustrate some common multi-response hypotheses, suppose that $p_1 = p_2 = p_3 = q = 3$ so that three variates are observed over three time points and that the number of groups g = 2. Then,

$$B = \begin{pmatrix} \beta_{10} & \beta_{11} & \beta_{12} & \theta_{10} & \theta_{11} & \theta_{12} & \xi_{10} & \xi_{11} & \xi_{12} \\ \beta_{20} & \beta_{21} & \beta_{22} & \theta_{20} & \theta_{21} & \theta_{22} & \xi_{20} & \xi_{21} & \xi_{22} \end{pmatrix} \tag{51}$$

$$W = \begin{pmatrix} \underset{\sim}{1}_{J_1} & 0 \\ 0 & \underset{\sim}{1}_{J_2} \end{pmatrix} \tag{52}$$

and

$$
(53) \qquad P = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 4 & 9 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 2 & 3 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 4 & 9 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 9
\end{pmatrix}.
$$

or

$$
(54) \qquad P = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1
\end{pmatrix}
$$

or

$$
(55) \; P = \begin{pmatrix}
.333333 & .333333 & .333333 & 0 & 0 & 0 & 0 & 0 & 0 \\
-.408248 & .000000 & .408248 & 0 & 0 & 0 & 0 & 0 & 0 \\
.235702 & -.471405 & .235702 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & .333333 & .333333 & .333333 & 0 & 0 & 0 \\
0 & 0 & 0 & -.408248 & .000000 & .408248 & 0 & 0 & 0 \\
0 & 0 & 0 & .235702 & -.471405 & .235702 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & .333333 & .333333 & .333333 \\
0 & 0 & 0 & 0 & 0 & 0 & -.408248 & .000000 & .408248 \\
0 & 0 & 0 & 0 & 0 & 0 & .235702 & -.471405 & .235702
\end{pmatrix}
$$

The first hypothesis of interest is whether the regression functions are parallel for all variables simultaneously:

(56) $$H_p: \begin{pmatrix} \beta_{11} \\ \beta_{12} \\ \theta_{11} \\ \theta_{12} \\ \xi_{11} \\ \xi_{12} \end{pmatrix} = \begin{pmatrix} \beta_{21} \\ \beta_{22} \\ \theta_{21} \\ \theta_{22} \\ \xi_{21} \\ \xi_{22} \end{pmatrix}$$

This is tested by defining C' and A as

(57) $$C' = (1 \quad -1) \text{ and } A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Given parallelism, to test for coincidence,

(58)
$$H_C: \begin{pmatrix} \beta_{10} \\ \beta_{11} \\ \beta_{12} \\ \theta_{10} \\ \theta_{11} \\ \theta_{12} \\ \xi_{10} \\ \xi_{11} \\ \xi_{12} \end{pmatrix} = \begin{pmatrix} \beta_{20} \\ \beta_{21} \\ \beta_{22} \\ \theta_{20} \\ \theta_{21} \\ \theta_{22} \\ \xi_{20} \\ \xi_{21} \\ \xi_{22} \end{pmatrix}$$

the matrices C' and A are given by

(59)
$$C' = (1 \quad -1) \quad A = \begin{pmatrix} I_3 & 0 & 0 \\ 0 & I_3 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

Other hypotheses such as

(60)
$$H_G: \begin{pmatrix} \beta_{11} \\ \beta_{21} \\ \theta_{11} \\ \theta_{21} \\ \xi_{11} \\ \xi_{21} \end{pmatrix} = \begin{pmatrix} \beta_{12} \\ \beta_{22} \\ \theta_{12} \\ \theta_{22} \\ \xi_{12} \\ \xi_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

or

(61)
$$H_T: \begin{pmatrix} \beta_{11}+\beta_{21} \\ \theta_{11}+\theta_{21} \\ \xi_{11}+\xi_{21} \end{pmatrix} = \begin{pmatrix} \beta_{12}+\beta_{22} \\ \theta_{12}+\theta_{22} \\ \xi_{12}+\xi_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

given parallelism, may also be tested.

To test that the best polynomial for all variables is linear, we would use the test matrices

$$C' = I_2 \text{ and } A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{62}$$

Given that this hypothesis is tenable, we have that $p_1 < q$, $p_2 < q$ and $p_3 < q$. Hence,

$$B = \begin{pmatrix} \beta_{10} & \beta_{11} & \theta_{10} & \theta_{11} & \xi_{10} & \xi_{11} \\ \beta_{20} & \beta_{21} & \theta_{20} & \theta_{21} & \xi_{20} & \xi_{21} \end{pmatrix}. \tag{63}$$

and

$$P = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 3 \end{pmatrix} \tag{64}$$

We may now test hypotheses about the elements of the (2×6) matrix B; however, the selection of the weight matrix G affects the analysis. Rao (1967) and Grizzle and Allen (1969) point out that weighting does not always produce shorter confidence intervals. By setting $G = I$, all the covariates are ignored and by using the Rao-Khatri model the covariates are applied to all other variables in the model. That is, the $3q-p_1-p_2-p_3$ covariates are used to adjust all the remaining variables simultaneously which is exactly what is done in multivariate analysis of covariance designs. To use different covariates with different sets of dependent variables, the generalized growth curve multivariate model discussed by Kleinbaum (1970, 1973b) is used.

To analyze profile data for measurements arranged as in Figure 5, the unrestricted full rank linear model given in (9) is used. Letting $p_1 = p_2 = p_3 = q = 3$ for the arrangement of population parameters shown in Table 16, we consider some hypotheses which might be of interest for profile data.

Table 16. Means for Multi-response Profile Data

		Variables								
		1			2			3		
Conditions		C_1	C_2	C_3	C_1	C_2	C_3	C_1	C_2	C_3
Treatments	A_1	μ_{11}	μ_{12}	μ_{13}	μ_{14}	μ_{15}	μ_{16}	μ_{17}	μ_{18}	μ_{19}
	A_2	μ_{21}	μ_{22}	μ_{23}	μ_{24}	μ_{25}	μ_{26}	μ_{27}	μ_{28}	μ_{29}

The first hypothesis of interest for profile data is whether the profiles for each variable are parallel. That is, is there an interaction between conditions and treatments. The hypothesis may be stated as

(65) $$H_{(AC)*}: \begin{pmatrix} \mu_{11}-\mu_{12} \\ \mu_{12}-\mu_{13} \\ \mu_{14}-\mu_{15} \\ \mu_{15}-\mu_{16} \\ \mu_{17}-\mu_{18} \\ \mu_{18}-\mu_{19} \end{pmatrix} = \begin{pmatrix} \mu_{21}-\mu_{22} \\ \mu_{22}-\mu_{23} \\ \mu_{24}-\mu_{25} \\ \mu_{25}-\mu_{26} \\ \mu_{27}-\mu_{28} \\ \mu_{28}-\mu_{29} \end{pmatrix}$$

The matrices C' and A to test $H_{(AC)*}$ are

(66) $$C'_{(AC)*} = (1 \quad -1) \text{ and } A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

To test for differences in treatments, H_{A*}, where H_{A*} is

(67) $$H_{A*}: \begin{pmatrix} \mu_{11} \\ \mu_{12} \\ \mu_{13} \\ \mu_{14} \\ \mu_{15} \\ \mu_{16} \\ \mu_{17} \\ \mu_{18} \\ \mu_{19} \end{pmatrix} = \begin{pmatrix} \mu_{21} \\ \mu_{22} \\ \mu_{23} \\ \mu_{24} \\ \mu_{25} \\ \mu_{26} \\ \mu_{27} \\ \mu_{28} \\ \mu_{29} \end{pmatrix}$$

the matrices

$$C'_{A*} = (1 \quad -1) \text{ and } A = I_9 \tag{68}$$

are constructed. For differences in conditions,

$$H_{C*}: \begin{pmatrix} \mu_{11} \\ \mu_{21} \\ \mu_{14} \\ \mu_{24} \\ \mu_{17} \\ \mu_{27} \end{pmatrix} = \begin{pmatrix} \mu_{12} \\ \mu_{22} \\ \mu_{15} \\ \mu_{25} \\ \mu_{18} \\ \mu_{28} \end{pmatrix} = \begin{pmatrix} \mu_{13} \\ \mu_{23} \\ \mu_{16} \\ \mu_{26} \\ \mu_{19} \\ \mu_{29} \end{pmatrix} \tag{69}$$

the test matrices are

$$C'_{C*} = I_2 \text{ and } A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{70}$$

Given parallelism, tests for differences between the two treatments and among conditions are written as

(71)

$$A: \begin{pmatrix} \sum_{j=1}^{3} \mu_{1j}/3 \\ \sum_{j=4}^{6} \mu_{1j}/3 \\ \sum_{j=7}^{9} \mu_{1j}/3 \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{3} \mu_{2j}/3 \\ \sum_{j=4}^{6} \mu_{2j}/3 \\ \sum_{j=7}^{9} \mu_{2j}/3 \end{pmatrix}$$

and

(72)

$$C: \begin{pmatrix} \sum_{i=1}^{2} \mu_{i1}/2 \\ \sum_{i=1}^{2} \mu_{i4}/2 \\ \sum_{i=1}^{2} \mu_{i7}/2 \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{2} \mu_{i2}/2 \\ \sum_{i=1}^{2} \mu_{i5}/2 \\ \sum_{i=1}^{2} \mu_{i8}/2 \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{2} \mu_{i3}/2 \\ \sum_{i=1}^{2} \mu_{i6}/2 \\ \sum_{i=1}^{2} \mu_{i9}/2 \end{pmatrix}$$

respectively. Hypothesis test matrices to test hypotheses A and C become

$$C_A' = (1\ -1) \quad A = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

and

$$(73) \qquad C_C' = \left(\frac{1}{2}\ \frac{1}{2}\right)\ A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{pmatrix}$$

Provided the post matrix A, for hypotheses stated as $C'UA = 0$, is normalized so that $A'A = I$, multivariate mixed model multivariate criteria are immediately obtained from the multivariate approach for the hypotheses A, C, and AC. This is not the case for the hypotheses A*, C* and (AC)* as we showed for split-plot designs, Timm and Carlson (1973). If the post matrix for testing the parallelism hypothesis is normalized, the multivariate mixed model hypothesis AC is immediately recovered from the multivariate test.

To illustrate the procedures discussed in this section, data provided by Dr. Tom Zullo in the School of Dental Medicine at the University of Pittsburgh, displayed in Table 17 are used.

Table 17. Individual Measurements Utilized to Assess the Changes in the Vertical Position of the Mandible at Three Time Points of Activator Treatment

Group	Subject Number	SOr-Me (mm)			ANS-Me (mm)			Pal-MP angle (degrees)		
		1	2	3	1	2	3	1	2	3
T_1	1	117.0	117.5	118.5	59.0	59.0	60.0	10.5	16.5	16.5
	2	109.0	110.5	111.0	60.0	61.5	61.5	30.5	30.5	30.5
	3	117.0	120.0	120.5	60.0	61.5	62.0	23.5	23.5	23.5
	4	122.0	126.0	127.0	67.5	70.5	71.5	33.0	32.0	32.5
	5	116.0	118.5	119.5	61.5	62.5	63.5	24.5	24.5	24.5
	6	123.0	126.0	127.0	65.5	61.5	67.5	22.0	22.0	22.0
	7	130.5	132.0	134.5	68.5	69.5	71.0	33.0	32.5	32.0
	8	126.5	128.5	130.5	69.0	71.0	73.0	20.0	20.0	20.0
	9	113.0	116.5	118.0	58.0	59.0	60.5	25.0	25.0	24.5
	Means	119.33	121.72	122.94	63.22	64.00	65.61	24.67	25.17	25.11
T_2	1	128.0	129.0	131.5	67.0	67.5	69.0	24.0	24.0	24.0
	2	116.5	120.0	121.5	63.5	65.0	66.0	28.5	29.5	29.5
	3	121.5	125.5	127.0	64.5	67.5	69.0	26.5	27.0	27.0
	4	109.5	112.0	114.0	54.0	55.5	57.0	18.0	18.5	19.0
	5	133.0	136.0	137.5	72.0	73.5	75.5	34.5	34.5	34.5
	6	120.0	124.5	126.0	62.5	65.0	66.0	26.0	26.0	26.0
	7	129.5	133.5	134.5	65.0	68.0	69.0	18.5	18.5	18.5
	8	122.0	124.0	125.5	64.5	65.5	66.0	18.5	18.5	18.5
	9	125.0	127.0	128.0	65.5	66.5	67.0	21.5	21.5	21.6
	Means	122.78	125.72	127.28	64.28	66.00	67.17	24.00	24.22	24.29

From the mean plots of the data in Table 17 for each group and variable, Figure 6, it appears that the growth curves for the three variables are at least linear. Some other questions of interest for the data include:

(1) Are the growth curves for the two groups parallel for one or more variables?

(2) If we have parallel growth curves, for some variables, are they coincident?

(3) What are the confidence band(s) for the expected growth curve(s)?

Depending on whether we take $p = q = 3$ when analyzing the data in Table 17, the procedure used to answer questions (1), (2) and (3) will differ. For illustration purposes, we will demonstrate both techniques using a program developed at the Educational Testing Service called ACOVSM, Jöreskog, van Thillo and Gruvaeus (1971).

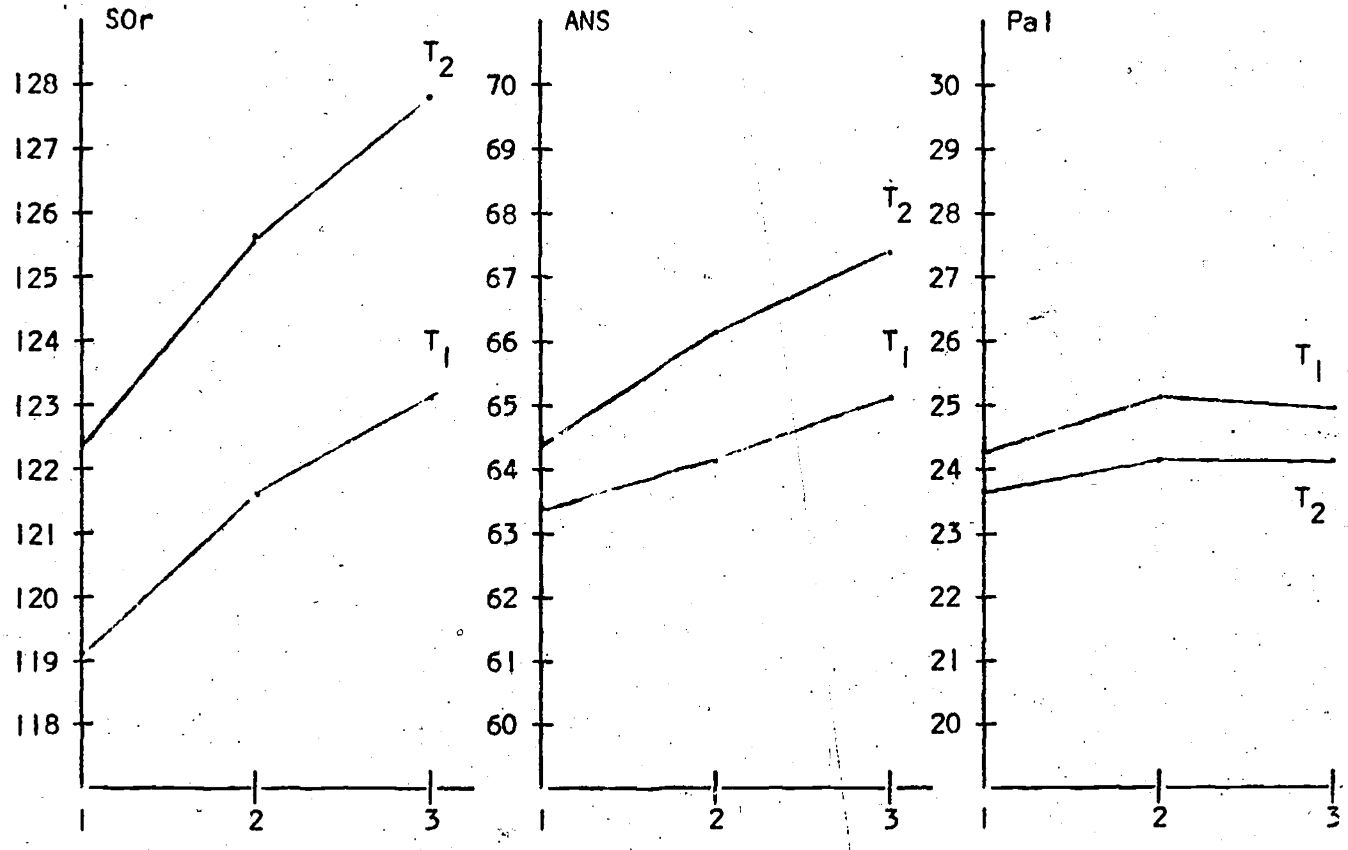

Figure 6. Mean Plots for Data in Table 17

Assuming that p = q = 3, the matrix B for the data in Table 17 is

$$B = \begin{pmatrix} \beta_{10} & \beta_{11} & \beta_{12} & \theta_{10} & \theta_{11} & \theta_{12} & \xi_{10} & \xi_{11} & \xi_{12} \\ \beta_{20} & \beta_{21} & \beta_{22} & \theta_{20} & \theta_{21} & \theta_{22} & \xi_{20} & \xi_{21} & \xi_{22} \end{pmatrix}$$

With P defined in (53), B is estimated by

$$\hat{B} = \begin{pmatrix} 115.778 & 4.139 & -0.583 & 63.278 & -0.472 & 0.417 & 23.611 & 1.333 & -0.278 \\ 118.444 & 5.028 & -0.694 & 62.000 & 2.556 & -0.278 & 23.622 & 0.456 & -0.078 \end{pmatrix}$$

To test for parallelism

$$H_p: \begin{pmatrix} \beta_{11} \\ \beta_{12} \\ \theta_{11} \\ \theta_{12} \\ \xi_{11} \\ \xi_{12} \end{pmatrix} = \begin{pmatrix} \beta_{21} \\ \beta_{22} \\ \theta_{21} \\ \theta_{22} \\ \xi_{21} \\ \xi_{22} \end{pmatrix}$$

simultaneously for all variables, the matrices

$$C' = (1 \quad -1) \text{ and } A = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{pmatrix}$$

are used where

$$A_i = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ for } i=1, 2, 3$$

Wilks' Λ-criterion for the test is $\Lambda = 0.583$ and comparing Λ with $U^{0.05}_{(6,1,16)} =$ 0.426 using Wall's table, Wall (1967), the parallelism test is not rejected. The p-value for the test is $\alpha_p = 0.3292$.

Given parallelism, we next test for coincidence,

$$H_C: \begin{pmatrix} \beta_{10} \\ \beta_{11} \\ \beta_{12} \\ \theta_{10} \\ \theta_{11} \\ \theta_{12} \\ \xi_{10} \\ \xi_{11} \\ \xi_{12} \end{pmatrix} = \begin{pmatrix} \beta_{20} \\ \beta_{21} \\ \beta_{22} \\ \theta_{20} \\ \theta_{21} \\ \theta_{22} \\ \xi_{20} \\ \xi_{21} \\ \xi_{22} \end{pmatrix}$$

again assuming $p = q$. For this test,

$$C' = (1 \quad -1) \text{ and } A = I_9$$

Computing Wilks' Λ-criterion, $\Lambda = 0.422$. Since tables for the U distribution are not available for $U^{\alpha} = U^{0.05}_{(9,1,16)}$, we may compute either Rao's multivariate F-statistic, $F = 1.216$ with 9 and 8 degrees of freedom or Bartlett's chi-squared statistic, $X^2 = 9.15$ with 9 degrees of freedom; both are approximations of the general U-distribution (see for example, Rao, 1973, p. 556 or Timm, 1974). The p-values for the two criteria are $\alpha_p = 0.3965$ and $\alpha_p = 0.3575$ respectively, indicating that we would not reject the coincidence hypothesis.

Treating the data in Table 17 as data obtained from a single group, we estimate the common regression function for all variables simultaneously with

$$\hat{B} = (117.111 \quad 4.583 \quad -0.639 \quad 62.639 \quad 1.041 \quad 0.069 \quad 23.617 \quad 0.894 \quad -0.178)$$

using either a restricted multivariate linear model or by pooling all the data into a single group. However, if this equation is taken as our final regression model we may have overfit one or more variables. This commonly occurs in experiments with five or more time points. Proceeding we test the hypothesis

$$H: \begin{pmatrix} \beta_{12} \\ \theta_{12} \\ \xi_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

using the matrices

$$C' = I \text{ and } \Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

to see if the mean trend associated with each variable given by the vector,

(121.056 123.722 125.111 63.750 65.000 66.389 24.333 24.694 24.700)

may be adequately represented using a linear model. Performing the test, $\Lambda = 0.438$ and comparing Λ with $U^{0.01}_{(3,1,17)} = 0.479$ the hypothesis is rejected. The p-value for the test is $\alpha_p = 0.0052$. Had we failed to reject the hypothesis, models for Zullo's data would have been represented by

$$y_{SOr} = 119.240 + 2.028t$$
$$y_{ANS} = 62.408 + 1.319t$$
$$y_{Pal} = 24.210 + 0.183t$$

if we set the weight matrix $G = I$. Confidence bands for linear equations would be obtained by setting

$C' = I$, $B = (\beta_{10} \;\; \beta_{11} \;\; \theta_{10} \;\; \theta_{11} \;\; \xi_{10} \;\; \xi_{11})$ and

$$A = \begin{pmatrix} 1 \\ t \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ or } A = \begin{pmatrix} 0 \\ 0 \\ 1 \\ t \\ 0 \\ 0 \end{pmatrix} \text{ or } A = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ t \end{pmatrix}$$

Bands for higher order polynomials follow similarly. If the growth curves were not coincident, we could also find confidence intervals for the difference between the two growth curves. The procedures are illustrated by Potthoff and Roy (1964), Khatri (1966), and Grizzle and Allen (1969).

Instead of analyzing Zullo's data with $p = q$, suppose that we decided a priori or through a statistical test that the regression model for each variable was linear. Then, $p < q$ and

$$B = \begin{pmatrix} \beta_{10} & \beta_{11} & \theta_{10} & \theta_{11} & \xi_{10} & \xi_{11} \\ \beta_{20} & \beta_{21} & \theta_{20} & \theta_{21} & \xi_{20} & \xi_{21} \end{pmatrix}$$

Using the Rao-Khatri model, with $G = S$, we test the coincidence hypothesis

$$H_C: \begin{pmatrix} \beta_{10} \\ \beta_{11} \\ \theta_{10} \\ \theta_{11} \\ \xi_{10} \\ \xi_{11} \end{pmatrix} = \begin{pmatrix} \beta_{20} \\ \beta_{21} \\ \theta_{20} \\ \theta_{21} \\ \xi_{20} \\ \xi_{21} \end{pmatrix}$$

by using the matrices.

$$C' = (1 \quad -1) \text{ and } A = I_6$$

For this test, $\Lambda = 0.440$ and comparing Λ with $U^{0.05}_{(6,1,13)} = 0.271$ we conclude that the growth curves for each group are coincident for all variables. The p-value for the test is $\alpha_p = 0.2300$. However, with $p < q$ and $G = S$, the models fit to each variable take the following form

$$y_{SOr} = 121.210 + 1.820t$$

$$y_{ANS} = 63.285 + 1.196t$$

$$y_{Pal} = 25.045 - 0.023t$$

which, as expected, do not agree with the models arrived at by taking $G = I$ since $p < q$.

Comparing the three regression models which may have been obtained using Zullo's data, the observed and predicted values for the models are displayed in Table 18.

Table 18. Regression Models

Observed Means	Predicted Means: Quadratic ($p = q$)	Linear ($p < q$, $G = I$)	Linear ($p < q$, $G = S$)
121.056	121.355	121.268	121.210
123.722	123.721	123.296	123.030
125.111	125.109	125.324	124.850
63.750	63.750	62.408	63.825
65.000	64.999	63.727	65.021
66.389	66.386	65.048	66.217
24.333	24.333	24.210	25.045
24.694	24.693	24.393	25.022
24.700	24.697	24.576	24.999

Using Wilks' Λ-criterion, we may construct $(1-\alpha)\%$ simultaneous confidence bands for each variable and each model. The general expressions for each band take the form

a) Potthoff-Roy

$$f(t) \pm \left[\left(\frac{1-U^{\alpha}}{U^{\alpha}} \right) (W'W)^{-1} (\underset{\sim}{f}' S_{e} \underset{\sim}{f}) \right]^{1/2}$$

with $S_e = Y'(I-W(W'W)^{-1}W')Y$ and $Y = Y_o G^{-1}P'(PG^{-1}P')^{-1}$

b) Rao-Khatri

$$f(t) \pm \left[\left(\frac{1-U^{\alpha}}{U^{\alpha}} \right) R(\underset{\sim}{f}'S_e\underset{\sim}{f}) \right]^{\frac{1}{2}}$$

with $S_e = (PS^{-1}P')^{-1}$

$$R = (W'W)^{-1}+[(W'W)^{-1}W'Y_o-\hat{B}P]S^{-1}Y_o'W(W'W)^{-1}$$

for a single group problem where $U^{\alpha} = U^{0.05}_{(9,1,17)} = 0.2392$ when $p = q$ or $p < q$ and $G = I$ and $U^{\alpha} = U^{0.05}_{(6,1,13)} = 0.2714$ when $p < q$ and $G = S$ for our example.

Assuming Zullo's data in Table 17 was obtained at three experimental conditions rather than three time points, tests of differences between groups (67), differences among conditions (69), and interaction between groups and conditions (65) would be of primary interest for the multivariate observations observed at each condition. Alternatively, given that the interaction hypothesis is tenable, tests of differences between groups and differences among conditions, as defined in (71) and (72), may be tesetd. Using the program described in Timm and Carlson (1973) to analyze the data in Table 17, Wilks' Λ-criterion and multivariate F-tests for the hypotheses are displayed in Table 19.

Table 19. Multi-response Profile Analysis of Zullo's Data

Hypotheses	Wilks' Λ	DF	Mult-F	DF	p-value
(AC)*	0.583	(6, 1, 16)	1.311	(6, 11)	0.2392
A*	0.422	(9, 1, 16)	1.216	(9, 8)	0.3965
C*	0.026	(6, 2, 16)	9.459	(12, 22)	<.0001
A	0.884	(3, 1, 16)	0.613	(3, 14)	0.6176
C	0.034	(6, 1, 16)	52.437	(6, 11)	<.0001

As we mentioned previously, mixed model multivariate tests are obtained from the appropriately normalized multivariate hypotheses in Table 19. To see this, consider the hypothesis and error mean squares and products matrices for testing C; the matrices were obtained by normalizing the post matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{pmatrix}$$

for the hypothesis given in (72) when writing the hypothesis in the form $C'UA = 0$:

$$MSP_C = \begin{bmatrix} 148.028 & & & & & \\ -26.927 & 4.898 & & & & \\ 96.319 & -17.521 & 62.674 & & & \\ 2.927 & -0.532 & 1.904 & 0.058 & & \\ 13.383 & -2.434 & 8.708 & 0.265 & 1.210 & \\ -7.493 & 1.363 & -4.875 & -0.148 & -0.677 & 0.379 \end{bmatrix}$$

$$MSP_E = \begin{bmatrix} 0.606 & & & & & \\ -0.277 & 0.337 & & & & \\ 0.457 & -0.199 & 0.556 & & & \\ -0.042 & 0.089 & -0.202 & 1.148 & & \\ -0.425 & 0.161 & -0.279 & 0.055 & 1.163 & \\ 0.233 & -0.116 & 0.183 & -0.049 & -0.634 & 0.383 \end{bmatrix}$$

Averaging the "circled" diagonal elements of the above matrices, the MSP_C and MSP_E matrices for the multivariate mixed model test of C are obtained:

$$MSP_C = \begin{bmatrix} 76.463 & & \\ 47.894 & 31.366 & \\ 7.373 & 4.280 & 0.795 \end{bmatrix} \qquad MSP_E = \begin{bmatrix} 0.472 & & \\ 0.273 & 0.852 & \\ -0.271 & -0.164 & 0.773 \end{bmatrix}$$

The degrees of freedom associated with the multivariate mixed model matrices are ν_h^* and ν_e^*; obtained from the formula

$$\nu_h^* = \nu_h \cdot R(A)/p = 1 \cdot 6/3 = 2$$
$$\nu_e^* = \nu_e \cdot R(A)/p = 16 \cdot 6/3 = 32$$

where p denotes the number of variables, R(A) is the rank of the post matrix A, and ν_h and ν_e are the hypothesis and error degrees of freedom associated with Wilks' Λ-criterion for testing C as displayed in Table 18. Wilks' Λ-criterion for testing C using the multivariate mixed model is $\Lambda = 0.0605$

which is compared to $U^{0.05}_{(3,2,32)} = 0.663$ or using the multivariate F-criterion, $F(6,60) = 30.64$, which we compare with $F^{0.05}_{(6,60)} = 3.12$. The p-value for the test is less than 0.0001.

Analyzing the data a variable at a time using three univariate mixed model split-plot designs, the univariate F-ratios for testing C are immediately obtained from the multivariate mixed model analysis. The univariate F-ratios are

Variables	F-value	p-value
SOr	76.463/0.472 = 162.1	< .0001
ANS	31.366/0.852 = 36.82	< .0001
Pal	0.795/0.773 = 1.03	0.3694

5. Summary

In this paper we have shown how multivariate models may be used to analyze repeated measures profile and growth curve data when univariate or multivariate mixed model assumptions are not tenable. This should not be taken to mean that a multivariate approach to the analysis of such designs should always be used or that it is always most appropriate. If the restrictive mixed model assumptions are tenable for a data set, we would always use the simplest model to analyze the data. For this reason, we have shown how one may recover standard mixed model tests from certain multivariate hypotheses. Procedures for testing whether mixed model assumptions are tenable are discussed in Timm (1974).

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